Finite Free Cumulants: Multiplicative Convolutions, Genus Expansion, and Infinitesimal Distributions

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About this talk

- This talk is mainly about finite free probability.

- Finite free probability arose from the work of Marcus, Spielman and Srivastava (2013-2016).

- Previous and subsequent work shows an interesting interaction between analytic theory of polynomials, free probability and random matrices.

Notation

- **(Real-rooted polynomials)** We will be interested in real-rooted polynomials:

$$p(x) = \prod_{i=1}^{d}(x - \alpha_i), \qquad \alpha_i \in \mathbb{R}, \forall i \in [d].$$

And their empirical root distribution

$$\mu_p := \frac{1}{d}\sum_{i=1}^{d}\delta_{\alpha_i}.$$

- Sometimes we will also talk about symmetric random matrices X_N and their (random) empirical spectral distribution by

$$\mu_{X_N} = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}.$$

Problem: Dynamics of root distributions after repeated differentiation

Setup: Let $(p_d)_{d=1}^{\infty}$ be a sequence of real-rooted polynomials, and assume that

$$\mu_{p_d} \to_w \nu.$$

for some compactly supported limiting probability distribution ν.

Question (Steinerberger 2018): For large d, how does the root distributions behave after repeated differentiation of the p_d?

To formalize this question, take $t \in (0,1)$ and study the asymptotic root distribution of the $p_d^{(\lfloor td \rfloor)}(x)$.

Problem: Dynamics of root distributions after repeated differentiation

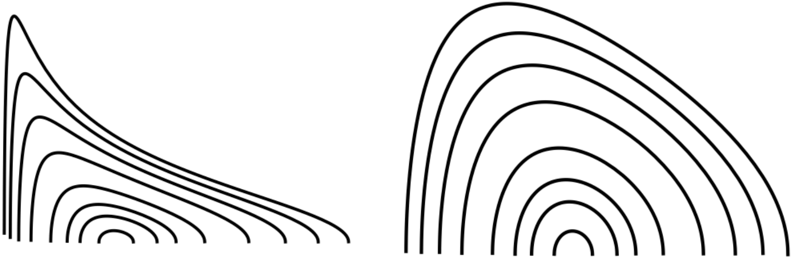


Image from *A Nonlocal Transport Equation Describing Roots of Polynomials Under Differentiation*, Steinerberger 2019.

Solution: fractional free convolution powers

- **Fractional free convolution powers.** Given $k \in \mathbb{Z}^+$ let $\nu^{\boxplus k} := \underbrace{\nu \boxplus \cdots \boxplus \nu}_{k \text{ times}}$.

  (Nica, Speicher 1996) There is a continuous interpolation $\nu^{\boxplus s}$ that is well defined for every $s \geq 1$.

- **(Steinerberger 2019-2020, Hoskins, Kabluchko 2020)** The root distributions of the $p_d^{(\lfloor td \rfloor)}(x)$, after proper re-scaling, converge in distribution to

$$\nu^{\boxplus \frac{1}{1-t}}.$$

- **(Arzimendi, JGV, Perales 2021)** Proof using finite free probability.

The additive convolution $\boxplus_d$

- **(Marcus, Spielman, Srivastava 2015)** Let $p(x)$ and $q(x)$ be degree d monic real-rooted polynomials. Let A and B be $d \times d$ Hermitian matrices with $p(x) = \det(x - A)$ and $q(x) = \det(x - B)$. Then

$$[p \boxplus_d q](x) := \mathbb{E}[\det(x - (A + U^* BU))],$$

where $U \sim \text{Haar}(\mathcal{U}(d))$.

- This convolution has been studied since 1922 (Walsh). If we write

$$p(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^p \qquad \text{and} \qquad q(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^q,$$

we have

$$[p \boxplus_d q](x) = \sum_{k=0}^{d} x^{d-k}(-1)^k \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} a_i^p a_j^q.$$

The multiplicative convolution $\boxtimes_d$

- **(Marcus, Spielman, Srivastava 2015)** Let $p(x)$ and $q(x)$ be degree d monic real-rooted polynomials, with $q(x)$ having only non-negative roots. Let A and B be $d \times d$ Hermitian matrices with $p(x) = \det(x - A)$ and $q(x) = \det(x - B)$. Then

$$[p \boxtimes_d q](x) := \mathbb{E}[\det(x - AU^* BU)],$$

  where $U \sim \text{Haar}(\mathcal{U}(d))$.

- This convolution has been studied since 1922 (Szegö). Given two polynomials of degree d

$$p(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^p \qquad \text{and} \qquad q(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^q,$$

  we have

$$[p \boxtimes_d q](x) = \sum_{k=0}^{d} x^{d-k}(-1)^k \frac{a_k^p a_k^q}{\binom{d}{k}}.$$

Results about $\boxplus_d$ and $\boxtimes_d$

- **(Walsh 1922)** If p and q are real-rooted then the same is true for $p \boxplus_d q$.
- **(Szegö 1922)** If p and q are real-rooted, and q has only non-negative roots, then $p \boxtimes_d q$ is real-rooted.
- **(Marcus 2016, Arizmendi, Perales 2016)** Let $(p_d)_{d=1}^{\infty}$ and $(q_d)_{d=1}^{\infty}$ be sequences of real rooted polynomials. If there are limiting (compactly supported) probability measures ν_p and ν_q such that

$$\mu_{p_d} \longrightarrow_w \nu_p \quad \text{and} \quad \mu_{q_d} \longrightarrow_w \nu_q$$

  then

$$\mu_{p_d \boxplus_d q_d} \longrightarrow_w \nu_p \boxplus \nu_q.$$

- **(Arizmendi, JGV, Perales 2021)** If the q_d have only non-negative roots

$$\mu_{p_d \boxtimes_d q_d} \longrightarrow_w \nu_p \boxtimes \nu_q$$

Problem

Setup: Let $(p_d)_{d=1}^{\infty}$ be a sequence of real-rooted polynomials, and let ν be a compactly supported probability measure such that

$$\mu_{p_d} \to_w \nu.$$

- **(Steinerberger 2019-2020, Hoskins, Kabluchko 2020)** The root distributions of the $p_d^{(\lfloor td \rfloor)}(x)$, after proper re-scaling, converge in distribution to
$$\nu^{\boxplus \frac{1}{1-t}}.$$

Proof (Arizmendi, JGV, Perales)

Step 1: *View $\boxtimes_d$ as a differential operator.* Let D denote differentiation with respect to x, that is $Df(x) = f'(x)$. We will write $\boxtimes_d$ in terms of xD.

Recall that if

$$p(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^p \qquad \text{and} \qquad q(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^q,$$

then

$$[p \boxtimes_d q](x) = \sum_{k=0}^{d} x^{d-k}(-1)^k \frac{a_k^p a_k^q}{\binom{d}{k}}.$$

- **(Mirabelli 2021)** If P and Q are polynomials such that $P(xD)(x-1)^d = p(x)$ and $Q(xD)(x-1)^d = q(x)$ then

$$[p \boxtimes_d q](x) = P(xD)Q(xD)(x-1)^d = P(xD)q(x) = Q(xD)p(x).$$

Proof (Arizmendi, JGV, Perales)

Step 2: *Write down $D^k p(x)$ using $\boxtimes_d$.* It is easy to show that for every k, there is a polynomial R with

$$R(xD) = x^k D^k.$$

For R as above

$$R(xD)(x-1)^d = x^k D^k (x-1)^d = (d)_k x^k (x-1)^{d-k},$$

So, from Step 1, for $r(x) = x^k (x-1)^{d-k}$ we get

$$p(x) \boxtimes_d r(x) = \frac{1}{(d)_k} x^k D^k p(x) = \frac{1}{(d)_k} x^k p^{(k)}(x).$$

Proof (Arizmendi, JGV, Perales)

Step 3: *Use that $\boxtimes_d$ converges to $\boxtimes$.* Recall that we have a sequence $(p_d)_{d=1}^\infty$ with $\mu_{p_d} \to_w \nu$ and we want to study the root distribution of $(p_d^{(\lfloor td \rfloor)})_{d=1}^\infty$.

For every d let $k_d := \lfloor td \rfloor$ and $r_d(x) := x^{k_d}(x-1)^{d-k_d}$. From Step 2 we know that

$$p_d(x) \boxtimes_d r_d(x) = \frac{1}{(d)_{k_d}} x^{k_d} p^{(k_d)}(x).$$

On the other hand it is clear that

$$\mu_{r_d} \longrightarrow_w t\delta_0 + (1-t)\delta_1.$$

Main new technical input: $\boxtimes_d \to_w \boxtimes$ implies

$$\mu_{p_d \boxtimes_d r_d} \longrightarrow_w \nu \boxtimes (t\delta_0 + (1-t)\delta_1).$$

And it is well known (Nica, Speicher 96) that $\nu \boxtimes (t\delta_0 + (1-t)\delta_1)$ is a dilation of

$$t\delta_0 + (1-t)\nu^{\boxplus \frac{1}{1-t}}.$$

Proof (Arizmendi, JGV, Perales)

- **Step 1:** *View $\boxtimes_d$ as a differential operator.* If Q is a polynomial satisfying that $Q(xD)(x-1)^d = q(x)$, then

$$p(x) \boxtimes_d q(x) = Q(xD)p(x).$$

- **Step 2:** *Write down D^k using $\boxtimes_d$.* If $r(x) := x^k(x-1)^{d-k}$ then

$$p(x) \boxtimes_d r(x) = \frac{1}{(d)_k} x^k D^k p(x) = \frac{1}{(d)_k} x^k p^{(k)}(x)$$

- **Step 3:** *Use that $\boxtimes_d$ converges to $\boxtimes$.* If $k := \lfloor td \rfloor$ and $r_d(x) := x^k(x-1)^{d-k}$ then

$$\mu_{x^k p_d^{(k)}} = \mu_{p_d \boxtimes_d r_d} \longrightarrow_w \nu \boxtimes (t\delta_0 + (1-t)\delta_1).$$

  And by the result of Nica and Speicher we can relate this measure to $\nu^{\boxplus \frac{1}{1-t}}$.

Cumulant approach (high level idea)

- **(Free probability)** We are interested in *free* convolutions between measures. The cumulant approach looks as follows:

$$\text{Measures} \longleftrightarrow \text{Moments} \longleftrightarrow \text{Free cumulants} \longleftrightarrow \text{Free convolutions}$$

- **(Finite free probability)** We are interested in *finite free* convolutions between polynomials. The cumulant approach looks as follows:

$$\text{Polynomials} \longleftrightarrow \text{Coefficients} \longleftrightarrow \text{F.F. cumulants} \longleftrightarrow \text{F.F. convolutions}$$

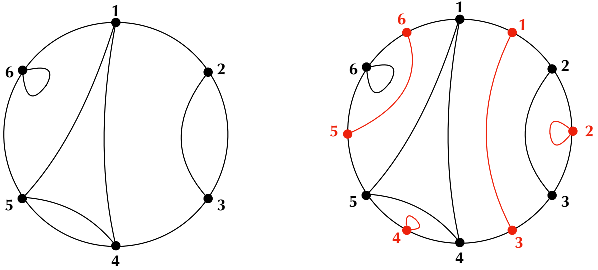
Remark: By the Newton relations the coefficients of a polynomial carry the same information as the moments of its empirical root distribution.

Notation

- **(Partitions)** We will denote the set of partiions and non-corssing partitions of $[n]$ by $\mathcal{P}(n)$ and $\mathcal{NC}(n)$ respectively.

- **(The Kreweras complement)** We will Kreweras complement by

$$Kr : \mathcal{NC}(n) \to \mathcal{NC}(n).$$



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- **(The Kreweras complement)** We will Kreweras complement by

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- **(Extension of sequences)** For sequences $(a_n)_{n=1}^{\infty}$ we will work with the natural extension $(a_\pi)_{\pi \in \mathcal{P}(n), n \geq 1}$. Where for every $n \in \mathbb{Z}^+$ and $\pi \in \mathcal{P}(n)$ we define

$$a_\pi := \prod_{V \in \pi} a_{|V|}.$$

E.g. If $\pi = \{\{1,4,5\},\{2,3\},\{6\}\}$ then

$$a_\pi = a_3 a_2 a_1.$$

Free cumulants

- **(Additive free convolution)** Given two probability measures ν_1 and ν_2 we have
$$\kappa_n(\nu_1 \boxplus \nu_2) = \kappa_n(\nu_1) + \kappa_n(\nu_2).$$

- **(Multiplicative free convolution)** Given two probability measures ν_1 and ν_2, Nica and Speicher (1996) showed that
$$\kappa_n(\nu_1 \boxtimes \nu_2) = \sum_{\pi \in \mathcal{NC}(n)} \kappa_\pi(\nu_1)\kappa_{Kr(\pi)}(\nu_2).$$

Finite free cumulants

(Arizmendi and Perales 2016):

- **(Finite free cumulants)** If p is a monic polynomial of degree d, the *order d finite free cumulants* of p:

$$\kappa_1^d(p), \ldots, \kappa_d^d(p),$$

are defined as function (a sum over $\mathcal{P}(n)$) of the coefficients $a_1^p, \ldots, a_d^p$:

$$\kappa_n^d(p) := \frac{(-d)^n}{d(n-1)!} \sum_{\pi \in \mathcal{P}(n)} (-1)^{|\pi|} \frac{N!_\pi a_\pi^p (|\pi|-1)!}{(d)_\pi}, \quad \text{for } n = 1, 2, \ldots, d.$$

Where $N!_\pi = \prod_{V \in \pi} |V|!$ and $(d)_\pi = \prod_{V \in \pi} (d)_{|V|}$.

Remark. Using Möbius inversion a formula for the a_i^p in terms of the $\kappa_n^d(p)$ can be derived.

Finite free cumulants

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- **(Finite free cumulants)** If p is a monic polynomial of degree d, the *order d finite free cumulants* of p:

$$\kappa_1^d(p), \dots, \kappa_d^d(p),$$

  are defined as function (a sum over $\mathcal{P}(n)$) of the coefficients $a_1^p, \dots, a_d^p$.

- **(Additivity)** If p and q are monic polynomials of degree d then

$$\kappa_n^d(p \boxplus_d q) = \kappa_n^d(p) + \kappa_n^d(q), \quad \text{for } n = 1, \dots, d.$$

- **(Convergence to free cumulants)** If $(p_d)_{d=1}^{\infty}$ satisfies that $\mu_{p_d} \to_w \nu$ then

$$\lim_{d\to\infty} \kappa_n^d(p_d) = \kappa_n(\nu) \quad \text{for all fixed } n.$$

Finite free cumulants

(Marcus 2016):

- **Hermite polynomials:** If $\hat{H}_d(x)$ is (a rescaling of) the d-th Hermite polynomial, then

$$\kappa_2^d(\hat{H}_d) = 1 \quad \text{and} \quad \kappa_n^d(\hat{H}_d) = 0 \quad \text{for all } n = 1, \ldots, d.$$

- **Laguerre polynomials:** If $\hat{L}_d^{(\lambda)}$ is (a rescaling and reparametrization) of the d-th generalized Laguerre polynomial then

$$\kappa_n^d(\hat{L}_d^{(\lambda)}) = \lambda \quad \text{for all } n = 1, \ldots, d.$$

Remark: $\mu_{\hat{H}_d} \longrightarrow_w \nu_{sc}$ and $\mu_{\hat{L}_d^{(\lambda)}} \longrightarrow_w \nu_{FP}^{(\lambda)}$.

What about $\boxtimes_d$?

- **(Arizmendi, JGV, Perales 2021)** Let p and q be monic polynomials of degree d. Then, the following formula holds:

$$\kappa_n^d(p \boxtimes_d q) = \frac{(-1)^{n-1}}{(n-1)!} \sum_{\substack{\sigma,\tau \in \mathcal{P}(n) \\ \sigma \vee \tau = 1_n}} d^{|\sigma|+|\tau|-n-1} \mu(0_n, \sigma)\mu(0_n, \tau)\kappa_\sigma^d(p)\kappa_\tau^d(q)$$

  where $\mu(0_n, \pi)$ is the Möbius function of $\mathcal{P}(n)$.

- One can show that the condition $\sigma \vee \tau = 1_n$ implies that $|\sigma| + |\tau| \leq n + 1$.

- Since we care about n fixed and d going to infinity, rewrite the above as

$$\kappa_n^d(p \boxtimes_d q) = \chi_0(p, q) + \frac{1}{d}\chi_1(p, q) + \cdots + \frac{1}{d^{n-1}}\chi_{n-1}(p, q).$$

- **Message:** By understanding the combinatorial structure of the χ_i one can understand many features about asymtoptics of polynomials.

Terms of order $\Theta(1)$

For the expansion $\kappa_n^d(p \boxtimes_d q) = \chi_0(p,q) + \frac{1}{d}\chi_1(p,q) + \cdots + \frac{1}{d^{n-1}}\chi_{n-1}(p,q)$, we proved there is a "better way" to write $\chi_0(p,q)$.

- **(Arizmendi, JGV, Perales 2021)** For any monic polynomials p, q of degree d

$$\kappa_n^d(p \boxtimes_d q) = \sum_{\pi \in \mathcal{NC}(n)} \kappa_\pi^d(p)\kappa_{Kr(\pi)}^d(q) + O\left(\frac{1}{d}\right)$$

- **(Arizmendi, JGV, Perales 2021)** In the setting

$$\mu_{p_d} \longrightarrow_w \nu_p \quad \text{and} \quad \mu_{q_d} \longrightarrow_w \nu_q$$

we have

$$\mu_{p_d \boxtimes_d q_d} \longrightarrow_w \nu_p \boxtimes \nu_q.$$

About the proof. Recall that $\kappa_n(\nu_p \boxtimes \nu_q) = \sum_{\pi \in \mathcal{NC}(n)} \kappa_\pi(\nu_p)\kappa_{Kr(\pi)}(\nu_q)$. We then show that $\lim_{d\to\infty} \kappa_n^d(p \boxtimes_d q) = \kappa_n(\nu_p \boxtimes \nu_q)$.

What about the terms of order $\Theta(1/d^k)$?

- The elementary enumerative approach becomes too complicated.

- We resorted to a more algebraic/topological approach by changing the sum over partitions for a sum over permutations.

- Using some results from the theory of maps and surfaces we prove that

$$\kappa_n^d(p \boxtimes_d q) = \chi_0(p,q) + \frac{1}{d}\chi_1(p,q) + \cdots + \frac{1}{d^{n-1}}\chi_{n-1}(p,q),$$

  is actually a topological expansion.

Going from partitions to permutations

Let S_n be the symmetric group on n elements. There is a very natural function

$$f : S_n \longrightarrow \mathcal{P}(n)$$

that consist of forgetting the ordering inside the cycles of a permutation.

E.g. If $\alpha = (1,4,5)(2,3)(6)$ then $f(\alpha) = \{\{1,4,5\},\{2,3\},\{6\}\}$.

Going from partitions to permutations

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Note that f preserves the type:

- **(Type)** Given $\pi \in \mathcal{P}(n)$ (resp. $\alpha \in S_n$), label its blocks $V_1, \ldots, V_k$ (resp. its cycles by $c_1, \ldots, c_k$) in such a way that $|V_1| \geq \cdots \geq |V_K|$.
  Then, the type of π is the tuple $[|V_1|, \cdots, |V_k|]$ (resp. $[|c_1|, \cdots, |c_k|]$).

  E.g. The type of $\pi = \{\{1,4,5\},\{2,3\},\{6\}\}$ is $[3,2,1]$.

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  Then, the type of π is the tuple $[|V_1|, \cdots, |V_k|]$ (resp. $[|c_1|, \cdots, |c_k|]$).

- **(Extensions of sequences)** Note that if $(a_n)_{n=1}^{\infty}$ is any sequence, then $a_\pi = a_\sigma$ whenever π and σ have the same type.

Notation. For $\alpha \in S_n$ use a_α to denote $a_{f(\alpha)}$ (this is well defined).

Going from partitions to permutations

Recall that

$$\kappa_n^d(p \boxtimes_d q) = \frac{(-1)^{n-1}}{(n-1)!} \sum_{\substack{\sigma,\tau \in \mathcal{P}(n) \\ \sigma \vee \tau = 1_n}} d^{|\sigma|+|\tau|-n-1} \mu(0_n,\sigma)\mu(0_n,\tau)\kappa_\sigma^d(p)\kappa_\tau^d(q).$$

- The value of each term only depends on the types of σ and τ.
- Recall that $\mu(0_n,\pi) = (-1)^{n-|\pi|}\prod_{V\in\pi}(|V|-1)!$ for any $\pi \in \mathcal{P}(n)$. *And hence* $|\mu(0_n,\pi)|$ *is the size of the inverse image of* π *under the function*

$$f : S_n \longrightarrow \mathcal{P}(n).$$

- One can rewrite the RHS as

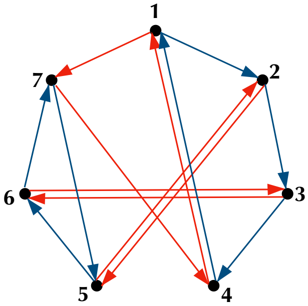
$$\frac{1}{(n-1)!} \sum_{\substack{\alpha,\beta \in S_n \\ f(\alpha)\vee f(\beta) = 1_n}} (-d)^{c(\alpha)+c(\beta)-n-1} \kappa_\alpha^d(p)\kappa_\beta^d(q)$$

Moreover $f(\alpha) \vee f(\beta) = 1_n$ is equivalent to $\langle \alpha, \beta \rangle$ acting transitively on $[n]$.

Theory of maps and surfaces

This theory goes back to Jacques (1968) and Cori (1975).

- Given two permutations $\alpha, \gamma \in S_n$, such that $\langle \alpha, \gamma \rangle$ acts transitively on $[n]$, we can construct a directed graph, denoted by $\mathcal{G}(\alpha|\gamma)$, whose vertex set is $[n]$ and where there is a directed edge between i and j if $\alpha(i) = j$ or $\gamma(i) = j$.

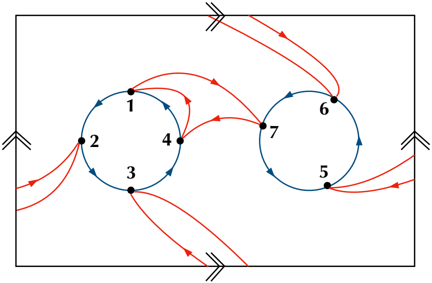


$\mathcal{G}(\alpha|\gamma)$ for $\alpha = (1, 7, 4)(2, 5)(3, 6)$ and $\gamma = (1, 2, 3, 4)(5, 6, 7)$.

Theory of maps and surfaces

This theory goes back to Jacques (1968) and Cori (1975).

- We are interested in embeddings of $\mathcal{G}(\alpha|\gamma)$ into closed oriented surfaces of some genus g:
  i) There are no crossings between the edges.
  ii) The interior of cycles coming from the permutations are homeomorphic to a disk.



Embedding of $\mathcal{G}(\alpha|\gamma)$ for $\alpha = (1, 7, 4)(2, 5)(3, 6)$ and $\gamma = (1, 2, 3, 4)(5, 6, 7)$.

Theory of maps and surfaces

This theory goes back to Jacques (1968) and Cori (1975).

- We are interested in embeddings of $\mathcal{G}(\alpha|\gamma)$ into closed oriented surfaces of some genus g:
  i) There are no crossings between the edges.
  ii) The (disjoint) interiors of cycles coming from the permutations are homeomorphic to a disk.

- **(Relative genus)** The genus of α relative to γ is defined to be the smallest g for which such an embedding exists.

- **(Euler's formula)** If g is the genus of α relative to γ then

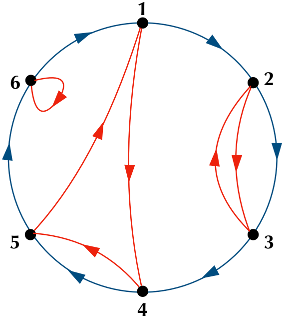
$$c(\alpha) + c(\alpha^{-1}\gamma) + c(\gamma) = n + 2(1 - g)$$

Non-crossing partitions as permutations

Let $\gamma_n = (1, \ldots, n) \in S_n$. Let $S_{NC}(n)$ be the set of $\alpha \in S_n$ of genus 0 relative to γ_n. Then there is a bijection

$$\psi : S_{NC}(n) \to \mathcal{NC}(n)$$

such that $\psi(\alpha^{-1}\gamma_n) = Kr(\psi(\alpha))$.



The embedding of $\mathcal{G}(\alpha|\gamma)$ when $\alpha = (1, 4, 5)(3, 2)(6)$ and $\gamma = \gamma_6$.

Proof of the formula for $\chi_0(p, q)$

- **(Arizmendi, JGV, Perales 2021)** For any monic polynomials p, q of degree d

$$\kappa_n^d(p \boxtimes_d q) = \sum_{\pi \in \mathcal{NC}(n)} \kappa_\pi^d(p)\kappa_{Kr(\pi)}^d(q) + O\left(\frac{1}{d}\right)$$

Step 1: *Use Euler's formula on the leading terms.* Recall that our formula for $\kappa_n^d(p \boxtimes_d q)$ can be rewritten as

$$\kappa_n^d(p \boxtimes_d q) = \frac{1}{(n-1)!} \sum_{\substack{\alpha,\beta \in S_n \\ f(\alpha)\vee f(\beta) = 1_n}} (-d)^{c(\alpha)+c(\beta)-n-1} \kappa_\alpha^d(p)\kappa_\beta^d(q)$$

The leading terms correspond to pairs α, β with $c(\alpha) + c(\beta) = n + 1$. Then

$$n + 1 + c(\alpha\beta) = c(\alpha) + c(\beta) + c(\alpha\beta) = n + 2(1 - g)$$

(by Euler's formula). **Hence:** $c(\alpha\beta) = 1$ and $g = 0$.

Proof of the formula for $\chi_0(p, q)$

Step 2: *Parametrize the sum using n-cycles.* From Step 1

$$\kappa_n^d(p \boxtimes_d q) = \frac{1}{(n-1)!} \sum_{\alpha,\beta \in S_n} \kappa_\alpha^d(p)\kappa_\beta^d(q) + O\left(\frac{1}{d}\right)$$

where the sum is over pairs with (i) $c(\alpha) + c(\beta) = n + 1$ and (ii) $c(\alpha\beta) = 1$.

Let $C_n = \{\gamma \in S_n : c(\gamma) = 1\}$. The sum in the RHS becomes:

$$\begin{aligned} &\sum_{\gamma \in C_n} \sum_{\substack{\alpha\beta = \gamma \\ c(\alpha)+c(\beta)=n+1}} \kappa_\alpha^d(p)\kappa_\beta^d(q) \\ =&\sum_{\gamma \in C_n} \sum_{c(\alpha)+c(\alpha^{-1}\gamma)=n+1} \kappa_\alpha^d(p)\kappa_{\alpha^{-1}\gamma}^d(q). \end{aligned}$$

Note that the α in the inner sum have relative genus 0 w.r.t. γ.

Proof of the formula for $\chi_0(p, q)$

Step 3: *Transport each γ to γ_n.* Let $\gamma_n := (1, \ldots, n)$. From Step 2 we have

$$\kappa_n^d(p \boxtimes_d q) = \frac{1}{(n-1)!} \sum_{\gamma \in C_n} \sum_{c(\alpha)+c(\alpha^{-1}\gamma)=n+1} \kappa_\alpha^d(p)\kappa_{\alpha^{-1}\gamma}^d(q) + O\left(\frac{1}{d}\right)$$

Idea: For every γ there is an inner automorphism of S_n that maps γ to γ_n, and inner automorphisms preserve types.

For every $\gamma \in C_n$ let $\tau_\gamma \in S_n$ be such that $\gamma_n = \tau_\gamma \gamma \tau_\gamma^{-1}$. Then if $\tilde{\alpha} := \tau_\gamma^{-1} \alpha \tau_\gamma$

$$\kappa_\alpha^d(p) = \kappa_{\tilde{\alpha}}^d(p), \quad \kappa_{\alpha^{-1}\gamma}^d(q) = \kappa_{\tilde{\alpha}^{-1}\gamma_n}^d(q), \quad \text{and} \quad c(\alpha)+c(\alpha^{-1}\gamma) = c(\tilde{\alpha})+c(\tilde{\alpha}^{-1}\gamma_n)$$

Hence, the RHS becomes

$$\frac{|C_n|}{(n-1)!} \sum_{c(\alpha)+c(\alpha^{-1}\gamma_n)=n+1} \kappa_\alpha^d(p)\kappa_{\alpha^{-1}\gamma_n}^d(q) + \left(\frac{1}{d}\right),$$

where $|C_n| = (n-1)!$ and $Kr(f(\alpha)) = f(\alpha^{-1}\gamma_n)$

Summary of the proof

Step 1: *Use Euler's formula on the leading terms:*

$$\kappa_n^d(p \boxtimes_d q) = \frac{1}{(n-1)!} \sum_{\alpha,\beta \in S_n} \kappa_\alpha^d(p)\kappa_\beta^d(q) + O\left(\frac{1}{d}\right)$$

where the sum is over pairs with (i) $c(\alpha) + c(\beta) = n + 1$ and (ii) $c(\alpha\beta) = 1$.

Step 2: *Parametrize the sum using n-cycles:*

$$\sum_{\gamma \in C_n} \sum_{c(\alpha)+c(\alpha^{-1}\gamma)=n+1} \kappa_\alpha^d(p)\kappa_{\alpha^{-1}\gamma}^d(q)$$

Step 3: *Transport each γ to γ_n:* By using inner automorphisms we showed that the sum corresponding to each γ has the same value. We chose γ_n as a representative to obtain:

$$\kappa_n^d(p \boxtimes_d q) = \sum_{c(\alpha)+c(\alpha^{-1}\gamma_n)=n+1} \kappa_\alpha^d(p)\kappa_{\alpha^{-1}\gamma_n}^d(q) + \left(\frac{1}{d}\right)$$

Terms of order $\Theta(1/d)$

For the expansion $\kappa_n^d(p \boxtimes_d q) = \chi_0(p,q) + \frac{1}{d}\chi_1(p,q) + \cdots + \frac{1}{d^{n-1}}\chi_{n-1}(p,q)$ the sum $\chi_1(p,q)$ can be written in terms of *annular non-crossing permutations.*

Terms of order $\Theta(1/d)$

Given r, s with $r + s = n$, let $\gamma_{r,s} = (1, \ldots, r)(r + 1, \ldots, r + s)$.

The set of annular non-crossing permutations $S_{NC}(r, s)$ is given by the $\alpha \in S_n$ of relative genus 0 with respect to $\gamma_{r,s}$ that satisfy that $\langle \alpha, \gamma_{r,s} \rangle$ acts transitively on $[n]$. Given $\alpha \in S_{NC}(r, s)$ denote $Kr_{r,s}(\alpha) := \alpha^{-1}\gamma_{r,s}$.

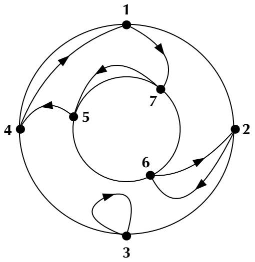


Diagram corresponding to $\alpha = (1, 7, 5, 4)(2, 6)(3)$ in $S_{NC}(4, 3)$.

Terms of order $\Theta(1/d)$

Recall

$$\kappa_n^d(p \boxtimes_d q) = \chi_0(p,q) + \frac{1}{d}\chi_1(p,q) + \cdots + \frac{1}{d^{n-1}}\chi_{n-1}(p,q) \tag{1}$$

- **(Arizmendi, JGV, Perales 2021)** For any monic polynomials $p(x), q(x)$ of degree d, and every n

$$\chi_1(p,q) = -\frac{n}{2} \sum_{\substack{r+s=n \\ \alpha \in S_{NC}(r,s)}} \frac{\kappa_\alpha^d(p)\kappa_{Kr_{r,s}(\alpha)}^d(q)}{rs}.$$

In general, the other terms in (1) can be written as sums over *multi-annular non-crossing permutations* (cf. higher-order freeness) in surfaces of different genera.

Infinitesimal distributions of polynomials

Setup (for random matrices): Let $(X_N)_{N=1}^{\infty}$ be a sequence of Hermitian random matirces and assume that the μ_{X_N} converge weakly (in expectation or in probability) to ν.

Question (for random matrices): How does μ_{X_N} fluctuate around ν?

- **(Infinitesimal distribution):** For every k let

$$m'_k := \lim_{N\to\infty} N \cdot \mathbb{E}\left[\int x^k d\mu_{X_N} - \int x^k d\nu \right].$$

  When the above limits exist, we say that $(X_N)_{N=1}^{\infty}$ has an infinitesimal distribution ν', where ν' is the signed measure with moments m'_k.

- **(GOE and Wishart)** There are explicit formulas for ν' when: (i) X_N is GOE (Johansson 1996), (ii) X_N is Wishart (Dumitriu and Edelman 2006).

Infinitesimal distributions of polynomials

Setup (for polynomials): Let $(p_d)_{d=1}^{\infty}$ be a sequence of real-rooted polynomials and assume that the μ_{p_d} converge weakly to ν.

Question (for random matrices): How does μ_{p_d} fluctuate around ν?

- **(Infinitesimal distribution):** For every k let

$$m'_k = \lim_{d\to\infty} d \cdot \left(\int x^k d\mu_{p_d} - \int x^k d\nu \right).$$

  When the above limits exist, we say that $(p_d)_{d=1}^{\infty}$ has an infinitesimal distribution ν', where ν' is the signed measure with moments m'_k.

- **(Hermite and Laguerre)** The infinitesimal distribution was computed by Dumitriu and Edelman in 2006.

Infinitesimal distributions

The formula for $\chi_1(p,q)$ allows to study infinitesimal distributions. A concrete result we can prove with this formula is:

- **(Arizmendi, JGV, Perales 2021)** Let ν be a compactly supported probability measure on $\mathbb{R}$. If $(p_d)_{d=1}^{\infty}$ is a sequence of real-rooted polynomials with

$$\kappa_n^d(p_d) = \kappa_n(\nu), \quad \forall d \quad \text{and} \quad n = 1, \ldots, d.$$

  Then the sequence $(p_d)_{d=1}^{\infty}$ has an infinitesimal distribution ν' given by

$$\nu' = \frac{1}{2}(M(\nu) - M(M(\nu))),$$

  where $M(\nu)$ is the Markov transform of ν.

- This can be applied directly when: (i) $\nu = \nu_{sc}$ and $p_d = \hat{H}_d$, (ii) $\nu = \nu_{FP}^{(\lambda)}$ and $p_d = \hat{L}_d^{(\lambda)}$, (iii) more in general when ν is a **free compound Poisson** and the p_d are "its finite free analogue".

Free Compound Poissons

Given a distribution ν let $m_n(\nu)$ denote its n-th moment. Given a polynomial p, let $m_n(p) := m_n(\mu_p)$.

- **(Free Compound Poisson)** The FCP distribution of parameter λ and jump distribution μ, is the measure ν defined by

$$\kappa_n(\nu) = \lambda m_n(\mu).$$

- **(F.F. compound Poisson)** Let $q(x)$ be a polynomial of degree d. The FFCP of parameter λ and jump $q(x)$ is the polynomial $p(x)$ of degree d satisfying

$$\kappa_n^d(p) = \lambda m_n(q)$$

- **(Compound Poissons via $\boxtimes_d$)** For any $q(x)$ of degree d

$$\lambda m_n(q) = \kappa_n^d(q \boxtimes_d \hat{L}^{(\lambda)}).$$

  Hence, if q is real-rooted, and $\lambda \geq 1$, the corresponding FFCP is real-rooted.

Formulas involving moments

- **(Arizmendi, JGV, Perales 2021)** For any p, q of degree d

$$m_n(p \boxtimes_d q) = \frac{(-1)^{n-1}}{(n-1)!} \sum_{\substack{\sigma,\tau \in \mathcal{P}(n) \\ \sigma \vee \tau = 1_n}} d^{|\sigma|+|\tau|-n-1} \mu(0_n, \sigma)\mu(0_n, \tau)\kappa^d_\sigma(p) m_\tau(q).$$

About the proof. Combine

$$m_n(p \boxtimes_d q) = \kappa^d_n((p \boxtimes_d q) \boxtimes_d \hat{L}^{(1)}) = \kappa^d_n(p \boxtimes_d (q \boxtimes_d \hat{L}^{(1)}))$$

with the cumulant-cumulant formula.

- When $q(x) = (x-1)^d$ the above formula recovers (Arizmendi, Perales 2016)

$$m_n(p) = \frac{(-1)^{n-1}}{(n-1)!} \sum_{\substack{\sigma,\tau \in \mathcal{P}(n) \\ \sigma \vee \tau = 1_n}} d^{|\sigma|+|\tau|-n-1} \mu(0_n, \sigma)\mu(0_n, \tau)\kappa^d_\sigma(p)$$

Infinitesimal distributions

- **(First two terms of the expansion)** For any p of degree d

$$m_n(p) = \sum_{\pi \in \mathcal{NC}(n)} \kappa_\pi^d(p) - \frac{n}{2d} \sum_{\substack{r+s=n \\ \alpha \in S_{NC}(r,s)}} \frac{\kappa_\alpha^d(p)}{rs} + O\left(\frac{1}{d^2}\right)$$

- **(Infinitesimal moments)** If ν is a probability measure and $\{p_d\}_{d=1}^\infty$ is a sequence satisfying $\kappa_n^d(p) = \kappa_n(\nu)$ for all $n = 1, \ldots d$, then

$$m_n(\nu) = \sum_{\pi \in \mathcal{NC}(n)} \kappa_\pi(\nu) = \sum_{\pi \in \mathcal{NC}(n)} \kappa_\pi^d(p)$$

Infinitesimal distributions

- **(First two terms of the expansion)** For any p of degree d

$$m_n(p) = \sum_{\pi \in \mathcal{NC}(n)} \kappa_\pi^d(p) - \frac{n}{2d} \sum_{\substack{r+s=n \\ \alpha \in S_{NC}(r,s)}} \frac{\kappa_\alpha^d(p)}{rs} + O\left(\frac{1}{d^2}\right)$$

- **(Infinitesimal moments)** If ν is a probability measure and $\{p_d\}_{d=1}^{\infty}$ is a sequence satisfying $\kappa_n^d(p) = \kappa_n(\nu)$ for all $n = 1, \dots d$, then

$$m_n(\nu) = \sum_{\pi \in \mathcal{NC}(n)} \kappa_\pi(\nu) = \sum_{\pi \in \mathcal{NC}(n)} \kappa_\pi^d(p).$$

Hence

$$m_n' = \lim_{d \to \infty} d(m_n(p) - m_n(\nu)) = \lim_{d \to \infty} -\frac{n}{2} \sum_{\substack{r+s=n \\ \alpha \in S_{NC}(r,s)}} \frac{\kappa_\alpha^d(p)}{rs} + O\left(\frac{1}{d}\right)$$

$$= -\frac{n}{2} \sum_{\substack{r+s=n \\ \alpha \in S_{NC}(r,s)}} \frac{\kappa_\alpha(\nu)}{rs}$$

Appearence of second order freeness

Using a formula from the theory of second order freeness we can compute $G_{inf}(z) := \sum_{n=1}^{\infty} m_n' z^{-n-1}$ when

$$m_n' = \lim_{d\to\infty} d(m_n(p) - m_n(\nu)) = -\frac{n}{2} \sum_{\substack{r+s=n \\ \alpha\in S_{NC}(r,s)}} \frac{\kappa_\alpha^d(p)}{rs}.$$

- **(Collins, Mingo, Śniady, Speicher 2007)**

$$G(z,w) = G'(z)G'(w)R(G(z),G(w)) + \frac{\partial^2}{\partial z \partial w} \log\left(\frac{G(z)-G(w)}{z-w}\right)$$

when the second order cumulants are zero

$$G(z,w) = \frac{\partial^2}{\partial z \partial w} \log\left(\frac{G(z)-G(w)}{z-w}\right)$$

Appearence of second order freeness

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- **(Collins, Mingo, Śniady, Speicher 2007)** Let $G_\nu(z) = \sum_{n=0}^{\infty} m_n z^{-n-1}$ and

  $$a_{r,s} := \sum_{\alpha\in S_{NC}(r,s)} \kappa_\alpha(\nu).$$

  Then the power series $G(z,w) := \sum_{r,s=1}^{\infty} a_{r,s} z^{-r-1} w^{-s-1}$ satisfies

  $$G(z,w) = \frac{\partial^2}{\partial z \partial w} \log\left(\frac{G_\nu(w) - G_\nu(z)}{w - z}\right)$$

Appearence of second order freeness

Using a formula from the theory of second order freeness we can compute $G_{inf}(z) := \sum_{n=1}^{\infty} m'_n z^{-n-1}$ when

$$m'_n = \lim_{d\to\infty} d(m_n(p) - m_n(\nu)) = -\frac{n}{2} \sum_{\substack{r+s=n \\ \alpha\in S_{NC}(r,s)}} \frac{\kappa_\alpha(\nu)}{rs}.$$

- **(Collins, Mingo, Śniady, Speicher 2007)** Let $G_\nu(z) = \sum_{n=0}^{\infty} m_n z^{-n-1}$ and $a_{r,s} := \sum_{\alpha\in S_{NC}(r,s)} \kappa_\alpha(\nu)$. Then $G(z,w) := \sum_{r,s=1}^{\infty} a_{r,s} z^{-r-1} w^{-s-1}$ satisfies

$$G(z,w) = \frac{\partial^2}{\partial z \partial w} \log\left(\frac{G_\nu(w) - G_\nu(z)}{w - z}\right)$$

- **(Arzimendi, JGV, Perales, 2021)** Using the above formula we were able to obtain

$$G_{inf}(z) = \frac{G''_\nu(z)}{2G'_\nu(z)} - \frac{G'_\nu(z)}{G_\nu(z)} \quad \text{and} \quad R_{inf}(z) = \frac{K''_\nu(z)}{2K'_\nu(z)} + \frac{1}{z}$$

Thank you!