Analytic theory of higher order freeness II

From the master relation to moment-cumulant formulas

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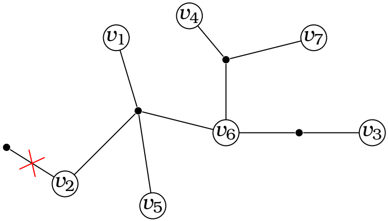
(Based on joint work with G. Borot, S. Charbonnier, F. Leid, S. Shadrin: arXiv:2112.12184)



Probabilistic Operator Algebra Seminar, UC Berkeley (online)

February 21, 2022

Outline

1. Moment-free cumulant relations: $M = G_{0,n} \leftrightarrow G^\vee_{0,n} = C$
   - Open question
   - First and second orders
   - Main result
2. Master relation: a universal duality?
   - Monotone Hurwitz numbers
3. Master relation $\Leftrightarrow \varphi = \zeta \circledast \varphi^\vee$, $\varphi^\vee = \kappa$
4. Master relation $\Leftrightarrow G_{g,n} \leftrightarrow G^\vee_{g,n}$
   - Fock space formalism
5. Origins of the master relation
   - Combinatorial maps and matrix models
   - From maps to free probability via matrix models
   - The origin of the master relation
   - Topological recursion and symplectic invariance

Outline

Recalling the open problem from Gaëtan's talk

$f : PS \to \mathbb{C}$ multiplicative function, with PS partitioned permutations.

$$f_{\ell_1,\ldots,\ell_n} := f(1_{\ell_1+\ldots+\ell_n}, \gamma_1 \cdots \gamma_n), \quad \gamma_i \text{ a cycle of length } \ell_i.$$

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- $\varphi \rightsquigarrow$ moments of a higher order probability space.
- $\kappa \rightsquigarrow$ free cumulants defined by $\kappa = \mu * \varphi$ ($\Leftrightarrow \varphi = \zeta * \kappa$, with $\zeta * \cdot \ \Leftrightarrow$ sum over non-crossing partitioned permutations).

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Encode $\varphi_{\ell_1,\dots,\ell_n}$ and $\kappa_{\ell_1,\dots,\ell_n}$ into the generating series:

$$n = 1: \quad M(x) := 1 + \sum_{\ell \geq 1} \varphi_\ell x^\ell, \quad C(w) := 1 + \sum_{\ell \geq 1} \kappa_\ell w^\ell.$$

Higher order:

$$M_n(x_1,\dots,x_n) := \sum_{\ell_1,\dots,\ell_n \geq 1} \varphi_{\ell_1,\dots,\ell_n} x_1^{\ell_1} \dots x_n^{\ell_n},$$

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Question: Functional relation between $M_n(x_1, \dots, x_n)$ and $C_n(w_1, \dots, w_n)$?

First and second orders

R-transform machinery:

- $n = 1$: (Voiculescu,'86)

$$C(xM(x)) = M.$$

Originally: Relation between the R-transform $R(w)$ and the Stieltjes transform $W(x)$, $C(w) = 1 + wR(w)$ and $W(x) = x^{-1}M(x^{-1})$.
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- $n \geq 3$? The number of types of $(1_{\ell_1+\ldots+\ell_n}, \gamma_1 \cdots \gamma_n)$—non-crossing partitioned permutations grows quickly $\Rightarrow$ their proof is hard to generalize.

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$\mathbf{n} = \mathbf{1}, \mathbf{2}$: (Borot, G-F ,'17) from combinatorics of fully simple maps.
$\mathbf{n} = \mathbf{3}$: (Borot, Charbonnier, G-F ,'21) for specific unitary invariant hermitian matrix models, from topological recursion.

Moment-free cumulant functional relations

- $\mathcal{G}_{0,n}(\mathbf{r}+1)$: set of bicoloured trees with white vertices labeled from 1 to n having valency $r_1+1,\ldots,r_n+1$, and without univalent black vertices.

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Theorem (Borot, Charbonnier, Leid, Shadrin, G-F, '21)

Let $x_i = w_i/C(w_i)$. For $n \geq 3$,

$$M_n(x_1,\ldots,x_n) = \sum_{r_1,\ldots,r_n\geq 0}\ \sum_{T\in\mathcal{G}_{0,n}(\mathbf{r}+1)} \left(\prod_{i=1}^{n} \mathrm{O}_{r_i}(w_i)\right) \prod_{I\in\mathcal{I}(T)}' C_{\#I}(w_I)\,.$$

- Weight per tree: $\mathcal{W}(T) := \prod'_{I\in\mathcal{I}(T)} C_{\#I}(w_I)$.
- $\prod' \rightsquigarrow C_2(w_i,w_j)$ should be replaced with $C_2(w_i,w_j) + \frac{w_iw_j}{(w_i-w_j)^2}$, if $i\neq j$.

Set of bicolored graphs

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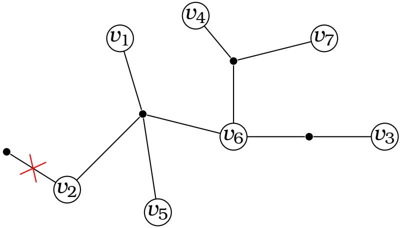
Example: n=7



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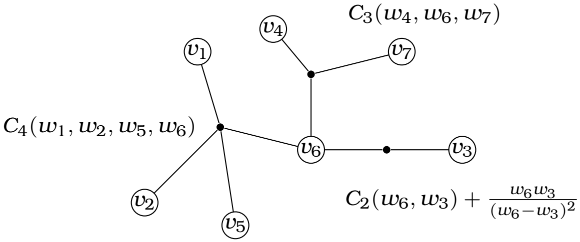
Example: $T \in \mathcal{G}_{0,7}(1,1,1,1,1,3,1)$



$$\mathcal{W}(T) = C_4(w_1,w_2,w_5,w_6)C_3(w_4,w_6,w_7)\Big(C_2(w_6,w_3) + \frac{w_6w_3}{(w_6-w_3)^2}\Big).$$

Finite sums and example

- $\mathcal{G}_{0,n}(\mathbf{r}+1)$: set of bicoloured trees with white vertices labeled from 1 to n having valency $r_1+1, \ldots, r_n+1$, and without univalent black vertices.

Remark

For n fixed, $\mathcal{G}_{0,n}(\mathbf{r}+1) \neq \emptyset$ only for finitely many $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{N}^n$.

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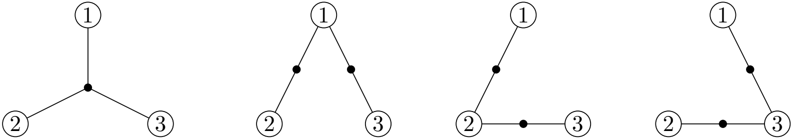

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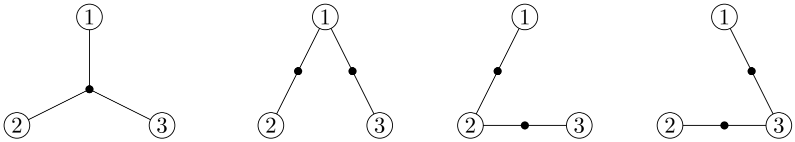

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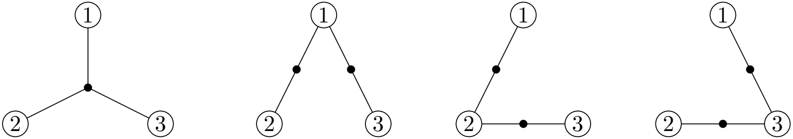

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Remarks $\Rightarrow$ The sums of the RHS of

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are finite.

Example: $n = 3$

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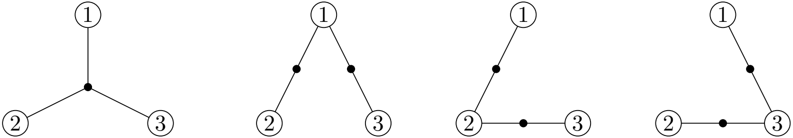

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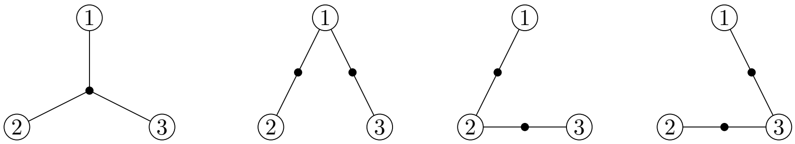

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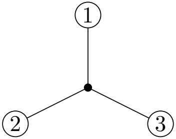

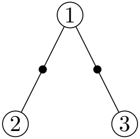

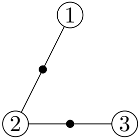

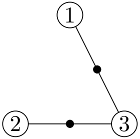

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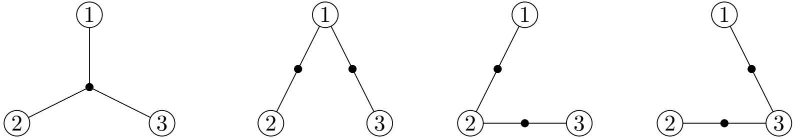

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$$M_3(x_1, x_2, x_3) = \frac{1}{\prod_{i=1}^3 C(w_i)x'(w_i)}\left(\mathcal{W}(T_0) + \sum_{i=1}^3 w_i \partial_{w_i} \frac{\mathcal{W}(T^{(i)})}{C(w_i)^2 x'(w_i)}\right),$$

with $\mathcal{W}(T^{(i)}) = \prod_{j \neq i}\left(C_2(w_i, w_j) + \frac{w_i w_j}{(w_i - w_j)^2}\right)$.

Beyond planar = beyond leading order (genus corrections)

To prove

$$\boxed{G_{0,n}(x_1,\ldots,x_n) := M_n(x_1,\ldots,x_n) \overset{\text{M-C}}{\leftrightarrow} G^\vee_{0,n}(w_1,\ldots,w_n) := C_n(w_1,\ldots,w_n),}$$

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we actually prove

$$\boxed{G_{g,n}(x_1,\ldots,x_n) \overset{\text{M-C}}{\leftrightarrow} G^\vee_{g,n}(w_1,\ldots,w_n)}$$

and specialize to $g = 0$.

Beyond planar = beyond leading order (genus corrections)

To prove

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$$\Rightarrow$$

Theory of moments and higher order free cumulants with genus corrections
(and a notion of (g,n)-freeness),
as we saw in Gaëtan's talk.

Outline

Double monotone Hurwitz numbers

$k, d \in \mathbb{Z}_{\geq 0}$, $\lambda, \mu \vdash d$.

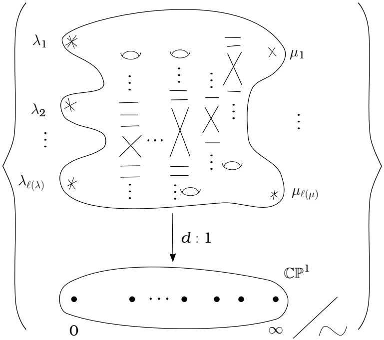
Definition

Double Hurwitz number $H_k(\lambda, \mu) \rightsquigarrow$ number of possibly disconnected coverings of the sphere with ramification profile

- λ over 0, μ over ∞,
- simply ramified over k points in $\mathbb{P}^1 \setminus \{0, \infty\}$,

weighted by $|\mathrm{Aut}|$.

$H_k(\lambda, \mu) := \#$



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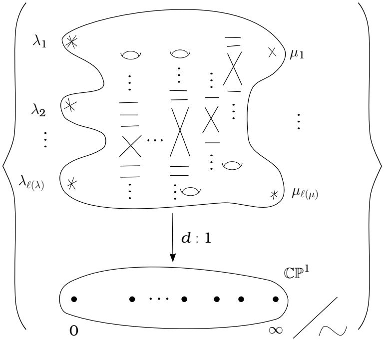

- $C_\lambda \rightsquigarrow$ Conjugacy class in $\mathfrak{S}_d$ of elements of cycle type $\lambda \vdash d$.

$$H_k(\lambda, \mu) = \frac{1}{d!} \left| \{ (\sigma, \tau_1, \ldots, \tau_k) \mid \sigma \in C_\lambda,\ \tau_i \in C_{(2,1\ldots,1)},\ \sigma\tau_1 \cdots \tau_k \in C_\mu \} \right| .$$

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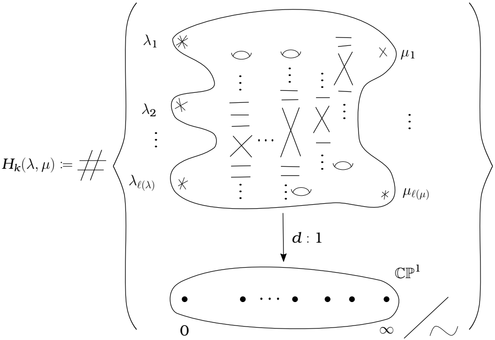

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Transpositions $\tau_i = (a_i\ b_i)$, with $a_i < b_i$, $i = 1, \ldots, k$:

- $b_i \leq b_{i+1} \rightsquigarrow$ *Weakly monotone:* $H_k^{\leq}(\lambda, \mu)$ (Goulden–Guay-Paquet–Novak,'11).
- $b_i < b_{i+1} \rightsquigarrow$ *Strictly monotone:* $H_k^{<}(\lambda, \mu)$.

$$\boxed{H^{<}(\lambda, \mu) = \sum_{k=0}^{d-1} H_k^{<}(\lambda, \mu) \hbar^k \in \mathbb{Q}[\hbar] \quad \text{and} \quad H^{\leq}(\lambda, \mu) = \sum_{k \geq 0} H_k^{\leq}(\lambda, \mu) \, (-\hbar)^k \in \mathbb{Q}[\![\hbar]\!].}$$

Topological partition functions and master relation

Fock space $\rightsquigarrow$ completion of the ring of symmetric polynomials with coefficients formal series in $\hbar$:

$$\mathcal{F}_R := R[\![p_1, p_2, p_3, \ldots]\!] , \qquad \mathcal{F}_{R,\hbar} := \mathcal{F}_R \otimes \mathbb{Q}(\!(\hbar)\!) .$$

- $\lambda \in \mathcal{Y} \rightsquigarrow$ Young diagrams. Consider $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}}$.
- $z(\lambda) = \prod_{i=1}^{\ell(\lambda)} \lambda_i \prod_{j>1} m_j(\lambda)!$, where $m_j(\lambda)$ is the number of $j's$ in λ.

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Topological partition function: $Z = e^F \in \mathcal{F}_{R,\hbar}$, $F = \sum_{g\geq 0} \hbar^{2g-2} F_g$, $F_g \in \mathcal{F}_R$.

$$Z = \exp\Bigg(\sum_{\substack{g\geq 0 \\ \lambda\in\mathcal{Y}}} \hbar^{2g-2} \frac{F_g(\lambda)}{z(\lambda)} p_\lambda \Bigg) = 1 + \sum_{\lambda\in\mathcal{Y}} \hbar^{-|\lambda|-\ell(\lambda)} Z(\lambda) p_\lambda .$$

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Dual formulation of the master relation:

$$(\star) \Leftrightarrow Z^\vee(\lambda) = z(\lambda) \sum_{\mu \vdash |\lambda|} H^{\leq}(\lambda, \mu) Z(\mu).$$

Multiplicative functions, correlators and strategy

Topological partition function $Z = e^F \leftrightarrow$ multiplicative function $\Phi_{Z,\hbar} \colon PS \to R[\![\hbar]\!]$:

- For $(\mathcal{A}, \alpha) \in PS(d)$:

$$\Phi_{Z,\hbar}(\mathcal{A}, \alpha) := \hbar^{d+\ell(\alpha)} \prod_{A \in \mathcal{A}} [p_{\alpha_{|A}}]\, F.$$

- For $\lambda \vdash d$:

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Strategy:

$$\Phi_{Z,\hbar} = \zeta_\hbar \circledast \Phi_{Z^\vee,\hbar}$$

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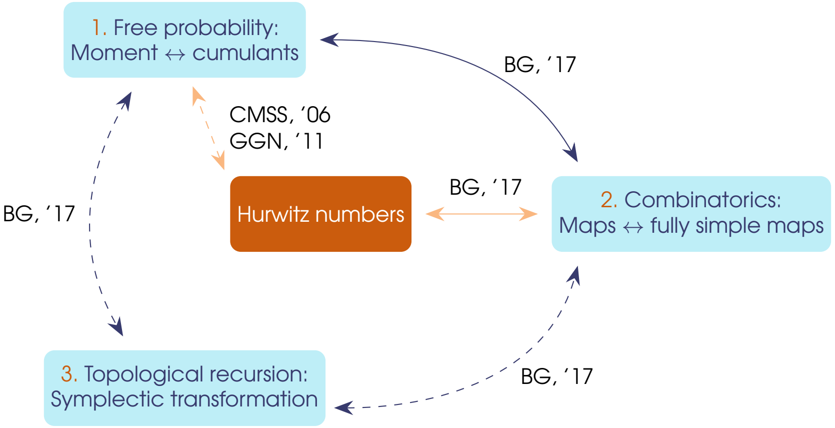
\end{tikzcd}$$

3 incarnations of the master relation: symplectic, simple and free



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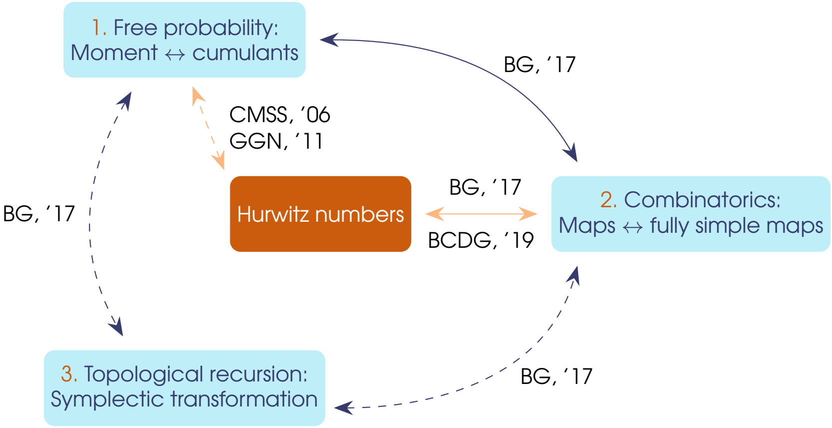

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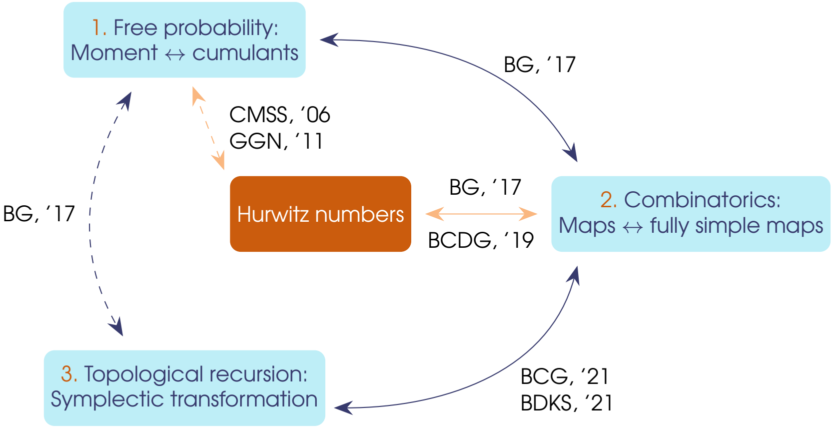

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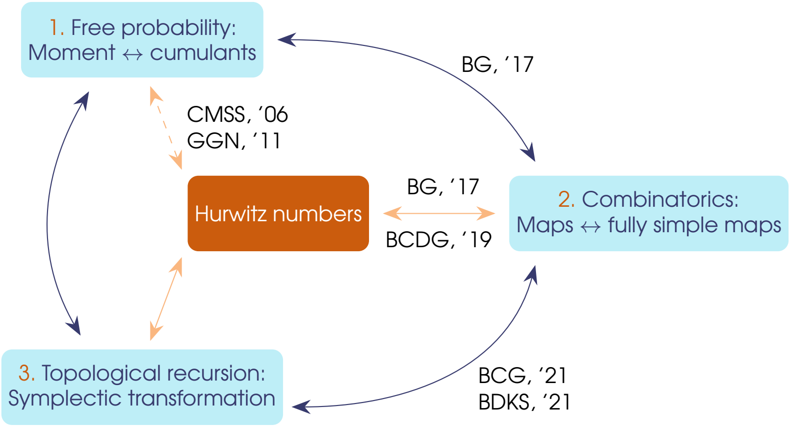

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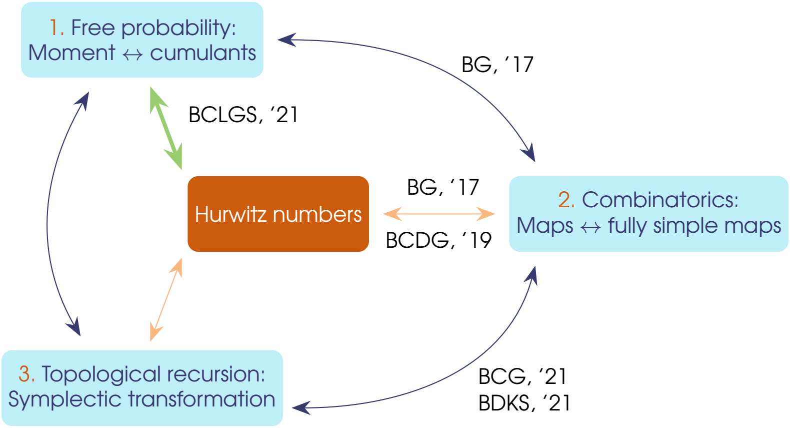

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Outline

1. Moment-free cumulant relations: $M = G_{0,n} \leftrightarrow G^\vee_{0,n} = C$
   - Open question
   - First and second orders
   - Main result
2. Master relation: a universal duality?
   - Monotone Hurwitz numbers
3. Master relation $\Leftrightarrow \varphi = \zeta \circledast \varphi^\vee$, $\varphi^\vee = \kappa$
4. Master relation $\Leftrightarrow G_{g,n} \leftrightarrow G^\vee_{g,n}$
   - Fock space formalism
5. Origins of the master relation
   - Combinatorial maps and matrix models
   - From maps to free probability via matrix models
   - The origin of the master relation
   - Topological recursion and symplectic invariance

Master relation is equivalent to convolution with $\zeta_\hbar$

$Z_1, Z_2 \rightsquigarrow$ two topological partition funtions.
$\Phi_{Z_1,\hbar}, \Phi_{Z_2,\hbar} \rightsquigarrow$ associated multiplicative functions on $PS(d)$.
$\Phi_{Z,\hbar}(\cdot) = \sum_g \hbar^{\deg} \varphi_Z(\cdot, g)$, with $\varphi_Z(\cdot, g)$ function on surfaced permutations.

Theorem (Borot, Charbonnier, Leid, Shadrin, G-F, '21)

Let $d \in \mathbb{Z}_{>0}$. The following 4 formulas are equivalent:

(i) $Z_1(\lambda) = z(\lambda) \sum_{\nu \vdash d} H^<(\lambda, \nu) Z_2(\nu)$, *for any* $\lambda \vdash d$;

(ii) $\Phi_{Z_1,\hbar} = \zeta_\hbar \circledast \Phi_{Z_2,\hbar}$;

(iii) $Z_2(\nu) = z(\nu) \sum_{\lambda \vdash d} H^\leq(\nu, \lambda) Z_1(\lambda)$, *for any* $\nu \vdash d$;

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Free single Hurwitz numbers:

$$H_r^{|}(\lambda, \mu) := \frac{1}{d!} |\{(\sigma, \alpha) \mid \sigma \in C_\lambda,\ \ell(\alpha) = d - k,\ \sigma\alpha \in C_\mu\}|.$$

Strictly monotone Hurwitz numbers:

$$H_r^{<}(\lambda, \mu) = \frac{1}{d!} |\{(\sigma, \tau_1, \ldots, \tau_k) \mid \sigma \in C_\lambda,\ \tau_i \in C_{(2,1,\ldots,1)}, \sigma\tau_1 \cdots \tau_k \in C_\mu,$$
$$\max \tau_1 < \cdots < \max \tau_k\}|.$$

$$\boxed{H_r^{<}(\lambda, \mu) = H_r^{|}(\lambda, \mu)}$$

Proof of the equivalence (i) $\Leftrightarrow$ (ii)

$$Z_1(\lambda) = \sum_{\substack{\mathcal{C} \in P(d) \\ \mathbf{0}_\lambda \leq \mathcal{C}}} \Phi_{Z_1,\hbar}(\mathcal{C}, \pi_\lambda) = \sum_{\substack{\mathcal{C} \in P(d) \\ \mathbf{0}_\lambda \leq \mathcal{C}}} (\zeta_\hbar \circledast \Phi_{Z_2,\hbar})(\mathcal{C}, \pi_\lambda)$$

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$$
\begin{aligned}
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&= \sum_{\substack{\mathcal{C}\in P(d)\\ \mathbf{0}_\lambda \le \mathcal{C}}} \sum_{(\mathbf{0}_\alpha,\alpha)\odot(\mathcal{B},\beta)=(\mathcal{C},\pi_\lambda)} \hbar^{|\alpha|}\, \Phi_{Z_2,\hbar}(\mathcal{B},\beta)
\end{aligned}
$$

Proof of the equivalence (i) $\Leftrightarrow$ (ii)

$$Z_1(\lambda) = \sum_{\substack{\mathcal{C}\in P(d)\\ \mathbf{0}_\lambda \le \mathcal{C}}} \Phi_{Z_1,\hbar}(\mathcal{C},\pi_\lambda) = \sum_{\substack{\mathcal{C}\in P(d)\\ \mathbf{0}_\lambda \le \mathcal{C}}} (\zeta_\hbar \circledast \Phi_{Z_2,\hbar})(\mathcal{C},\pi_\lambda)$$

$$= \sum_{\substack{\mathcal{C}\in P(d)\\ \mathbf{0}_\lambda \le \mathcal{C}}} \sum_{(\mathbf{0}_\alpha,\alpha)\odot(\mathcal{B},\beta)=(\mathcal{C},\pi_\lambda)} \hbar^{|\alpha|}\, \Phi_{Z_2,\hbar}(\mathcal{B},\beta)$$

$$= \sum_{\substack{\alpha,\beta\in S(d)\\ \alpha\circ\beta=\pi_\lambda}} \sum_{\substack{\mathcal{B}\in P(d)\\ \mathbf{0}_\beta \le \mathcal{B}}} \hbar^{|\alpha|} \Phi_{Z_2,\hbar}(\mathcal{B},\beta)$$

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Outline

Master relation in the Fock space

- $Z\mathbb{C}[\mathfrak{S}_d] \rightsquigarrow$ center of the group algebra of the symmetric group $\mathfrak{S}_d$.
- Basis (indexed by partitions $\lambda \vdash d$): $\hat{C}_\lambda = \sum_{\gamma \in C_\lambda} \gamma$.
- Jucys–Murphy elements $J_k = \sum_{i=1}^{k-1} (i\ k)$, $k \geq 2$.

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Identification (graded isomorphism) with the Fock space $\mathcal{F}_\mathbb{C}$ (completion of the ring of symmetric polynomials):

$$\begin{array}{cccc} \Psi : & \bigoplus_{d \geq 0} Z\mathbb{C}[\mathfrak{S}_d] & \simeq & \mathcal{F}_\mathbb{C} \\ & \hat{C}_\lambda & \mapsto & p_\lambda. \end{array}$$

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$$e_k(X_1, \ldots, X_n) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} X_{j_1} \cdots X_{j_k}$$ elementary symmetric polynomials.

$$\mathrm{D} := \prod_{k \geq 2} (1 + \hbar J_k) = \sum_{k \geq 0} \hbar^k e_k(J_2, J_3, \ldots) = \sum_{k \geq 0} \sum_{\substack{\tau_1, \ldots, \tau_k \\ (\max \tau_i)_{i=1}^k \\ \text{strictly increasing}}} \tau_1 \cdots \tau_k$$

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Under the identification Ψ, D acts on $\mathcal{F}_{\mathbb{C},\hbar}$:

Master relation: $Z(\lambda) = z(\lambda) \sum_{\nu \vdash d} H^<(\lambda, \nu) Z^\vee(\nu) \Leftrightarrow Z = \mathsf{D} Z^\vee$

Correlators in the Fock space $\mathcal{F}_{\mathbb{C},\hbar}$

$|\rangle := 1 \in \mathcal{F}_{\mathbb{C}}$ and $\langle| : \mathcal{F}_{\mathbb{C},\hbar} \to \mathbb{C}((\hbar))$ sending an element to the value at $p_{\lambda} = 0$.

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Heisenberg operators:

$$\mathrm{J}_k = \begin{cases} k\partial_{p_k} & \text{if } k > 0\,, \\ 0 & \text{if } k = 0\,, \\ p_{-k} & \text{if } k < 0\,, \end{cases}$$

and their generating series

$$\mathrm{J}(x) = \sum_{k>0} x^k \mathrm{J}_k\,,$$

$$\widetilde{J}(x) = \sum_{k \in \mathbb{Z}} x^k \mathrm{J}_k\,.$$

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$$\mathsf{F} = \sum_{\substack{n\geq 1 \\ g\geq 0}} \frac{\hbar^{2g-2}}{n!} \sum_{k_1,\dots,k_n>0} F_{g;k_1,\dots,k_n} \prod_{i=1}^{n} \frac{\mathsf{J}_{-k_i}}{k_i}\,.$$

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Connected n-point functions G_n and their shifted version $\widetilde{G}_n$:

$$\begin{aligned} G_n(x_1,\ldots,x_n) &= \hbar^{-1}\delta_{n,1} + \hbar^n \left\langle \left| \prod_{i=1}^{n} \mathsf{J}(x_i) \cdot e^{\mathsf{F}} \right| \right\rangle^{\circ} \\ &= \hbar^{-1}\delta_{n,1} + \sum_{g\geq 0} \hbar^{2g-2+n} G_{g,n}(x_1,\ldots,x_n), \end{aligned}$$

$$\widetilde{G}_n(x_1,\ldots,x_n) = \hbar^{-1}\delta_{n,1} + \hbar^n \left\langle \left| \prod_{i=1}^{n} \widetilde{\mathsf{J}}(x_i) \cdot e^{\mathsf{F}} \right| \right\rangle^{\circ} = G_n(x_1,\ldots,x_n) + \delta_{n,2}\,\frac{x_1 x_2}{(x_1 - x_2)^2}\,.$$

Idea of the proof

$$\boxed{Z = \mathsf{D}Z^\vee \Leftrightarrow G_{g,n} \xleftrightarrow{\text{M-C}} G^\vee_{g,n}}$$

- Substitute $\mathsf{D}^{-1}\,\widetilde{\mathsf{J}}(X)\,\mathsf{D}$ in $G_n(x_1,\ldots,x_n) = \hbar^{-1}\delta_{n,1} + \hbar^n \left\langle \middle| \prod_{i=1}^n \mathsf{J}(x_i)\cdot e^{\mathsf{F}} \middle| \right\rangle^\circ$, where $e^{\mathsf{F}}|\rangle = Z = \mathsf{D}Z^\vee$.

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- Using techniques from (Bychkov–Dunin-Barkowski–Kazarian–Shadrin, '21), transform $\mathsf{D}^{-1}\widetilde{\mathsf{J}}(x)\,\mathsf{D}$ into a sum over the graphs $\mathcal{G}_{g,n}$.

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- Using $\langle|\mathsf{D}^{-1} = \langle|$, deduce $G_{g,n} \xleftrightarrow{\text{M-C}} G^\vee_{g,n}$, which specialize to $G_{0,n} \xleftrightarrow{\text{M-C}} G^\vee_{0,n}$ for $g = 0$.

Outline

1. Moment-free cumulant relations: $M = G_{0,n} \leftrightarrow G^\vee_{0,n} = C$
   - Open question
   - First and second orders
   - Main result
2. Master relation: a universal duality?
   - Monotone Hurwitz numbers
3. Master relation $\Leftrightarrow \varphi = \zeta \circledast \varphi^\vee$, $\varphi^\vee = \kappa$
4. Master relation $\Leftrightarrow G_{g,n} \leftrightarrow G^\vee_{g,n}$
   - Fock space formalism
5. **Origins of the master relation**
   - Combinatorial maps and matrix models
   - From maps to free probability via matrix models
   - The origin of the master relation
   - Topological recursion and symplectic invariance

Maps and fully simple maps

Definition

A map of genus g and n *boundaries* is a connected graph Γ embedded into a closed oriented surface X of genus g such that

$$X \setminus \Gamma \cong \bigsqcup \mathbb{D} \quad \text{(\textit{faces}), with } n \text{ distinguished faces, (up to iso).}$$

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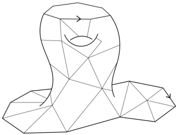

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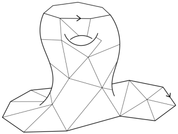

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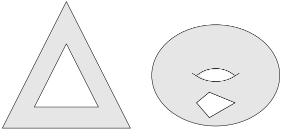

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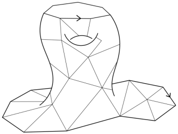

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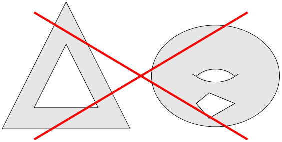

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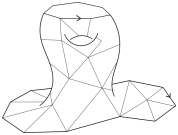

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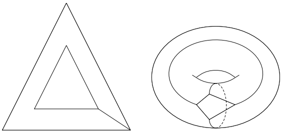

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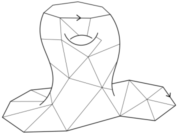

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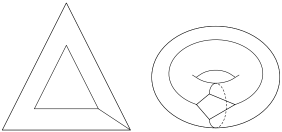
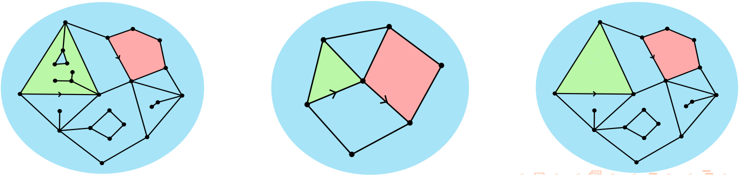


Simple: Boundaries are simple polygons.

Fully simple: Simple and pairwise disjoint boundaries.



Maps and formal hermitian matrix models

Generating series of maps of genus g and n boundaries of lengths $l_1, \ldots, l_n$:

$$\mathrm{Map}^{[g]}_{l_1,\ldots,l_n} := \sum_{\mathcal{M} \in \mathbb{M}^{[g]}_n(l_1,\ldots,l_n)} \prod_{f \in \mathrm{IFaces}(\mathcal{M})} t_{\mathrm{length}(f)} .$$

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Moments and classical cumulants:

$$\left\langle \prod_{i=1}^{n} \mathrm{Tr}\, M^{\ell_i} \right\rangle \quad \text{and} \quad c_n(\mathrm{Tr}\, M^{\ell_1}, \ldots, \mathrm{Tr}\, M^{\ell_n}).$$

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- $\gamma = (c_1\, c_2\, \ldots\, c_{\ell(\gamma)})$ cycle in $\mathfrak{S}_N \rightsquigarrow \mathcal{P}_\gamma(M) := \prod_{i=1}^{\ell(\gamma)} M_{c_i, \gamma(c_i)}$.

$$\left\langle \prod_{i=1}^{n} \mathcal{P}_{\gamma_i}(M) \right\rangle \quad \text{and} \quad c_n\big(\mathcal{P}_{\gamma_1}(M), \ldots, \mathcal{P}_{\gamma_n}(M)\big),$$

where γ_i are pairwise disjoint cycles of $\mathfrak{S}_N$ ($N \geq \sum_{i=1}^{n} \ell(\gamma_i)$).

From maps to free probability via matrix models

Free probability from matrix model:

$$\varphi_{\ell_1,\dots,\ell_n} = \lim_{N\to\infty} N^{n-2} c_n\big(\mathrm{Tr}\, M^{\ell_1}, \dots, \mathrm{Tr}\, M^{\ell_n}\big),$$

$$\kappa_{\ell_1,\dots,\ell_n} = \lim_{N\to\infty} N^{n-2+d} c_n\big(\mathcal{P}_{\gamma_1}(M), \dots, \mathcal{P}_{\gamma_n}(M)\big),\ d = \sum_{i=1}^{n} \ell_i.$$

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$$c_n\big(\mathrm{Tr}\, M^{\ell_1},\ldots,\mathrm{Tr}\, M^{\ell_n}\big) = \sum_{g\geq 0} N^{2-2g-n}\mathrm{Map}^{[g]}_{\ell_1,\ldots,\ell_n},$$

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Remark: For more general multi-tracial hermitian measures, stuffed maps.

From maps to free probability

$$\varphi_{\ell_1,\ldots,\ell_n} = \mathrm{Map}^{[0]}_{\ell_1,\ldots,\ell_n}, \quad \kappa_{\ell_1,\ldots,\ell_n} = \mathrm{FSMap}^{[0]}_{\ell_1,\ldots,\ell_n}.$$

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From maps to free probability (with genus corrections)

$$\varphi^{[g]}_{\ell_1,\ldots,\ell_n} = \mathrm{Map}^{[g]}_{\ell_1,\ldots,\ell_n}, \quad \kappa^{[g]}_{\ell_1,\ldots,\ell_n} = \mathrm{FSMap}^{[g]}_{\ell_1,\ldots,\ell_n}.$$

The origin of the master relation

$\lambda \vdash d$. $\mathrm{Map}^\bullet_\lambda$ and $\mathrm{FSMap}^\bullet_\lambda$ generating series of possibly disconnected maps with boundary lengths given by λ and with weight $N^{\chi(\mathcal{M})}$.

Theorem (Borot–G-F, '17, Borot–Charbonnier–Do–G-F, '19)

$$\mathrm{FSMap}^\bullet_\lambda = z(\mu) \sum_{\lambda \vdash d} H^{\leq}(\lambda,\mu)\big|_{\hbar = \frac{1}{N}} \mathrm{Map}^\bullet_\mu, \tag{1}$$

$$\mathrm{Map}^\bullet_\lambda = z(\lambda) \sum_{\mu \vdash d} H^{<}(\lambda,\mu)\big|_{\hbar = \frac{1}{N}} \mathrm{FSMap}^\bullet_\mu. \tag{2}$$

3 proofs:

- Via matrix models: Express

$$\mathrm{FSMap}^\bullet_\lambda = \Big\langle \mathcal{P}_\lambda(A) \Big\rangle = \Big\langle \prod_{i=1}^{n} \mathcal{P}_{\gamma_i}(A) \Big\rangle = \Big\langle \int_{\mathcal{U}_N} \mathcal{P}_\lambda(UAU^{-1}) \mathrm{d}U \Big\rangle$$

  in terms of the $\Big\langle \prod_{i=1}^n \mathrm{Tr}\, M^{\lambda_i} \Big\rangle$, using Weingarten calculus.
- 2 combinatorial proofs $\rightsquigarrow$ 1 via bijective combinatorics.

Proof via bijective combinatorics (joint work with G. Borot, S. Charbonnier and N. Do)

Definition

Dessin d'enfant $\rightsquigarrow$ map with each edge adjacent to one boundary face and one internal face. Boundary faces $\rightsquigarrow$ blue faces and internal faces $\rightsquigarrow$ red faces.

Proof via bijective combinatorics (joint work with G. Borot, S. Charbonnier and N. Do)

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$D_k(\lambda, \mu) \rightsquigarrow$ number of (possibly disconnected) dessins d'enfant with blue face degrees by λ and red face degrees by μ, and with k more edges than vertices.

$$D_k(\lambda, \mu) = z(\lambda) H^{|}_k(\lambda, \mu) = z(\lambda) H^{<}_k(\lambda, \mu).$$

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Idea: Construct a bijective function:

map $\longmapsto$ (fully simple map, dessin d'enfant)

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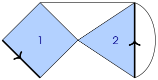
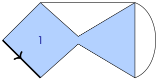
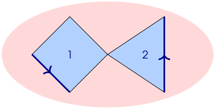


ordinary map

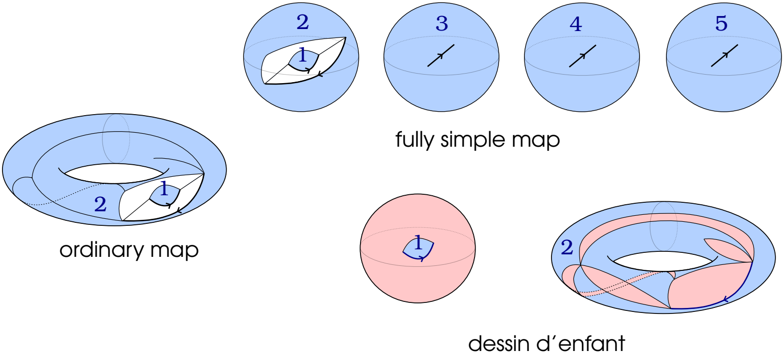


fully simple map



dessin d'enfant

Proof via bijective combinatorics (joint work with G. Borot, S. Charbonnier and N. Do)



Slogan: The fully simple map encodes the internal faces of the map while the dessin encodes how the boundaries of the map intersect.

Topological recursion (TR, Chekhov–Eynard–Orantin '04-'07)

Goal: "Count surfaces $S_{g,n}$ of genus g with n boundaries (topology (g,n))."

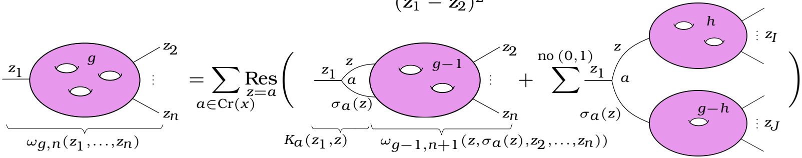
Spectral curve

TR : $\begin{cases} \Sigma \text{ Riemann surface} \\ x\colon \Sigma \to \mathbb{CP}^1 \\ \omega_{0,1} = y\,\mathrm{d}x \ \ 1\text{-form} \quad (\text{discs}) \\ \omega_{0,2} \ \ (1,1)\text{-form} \quad (\text{cylinders}) \end{cases}$ $\rightsquigarrow$ **recursion on** $|\chi(S_{g,n})| = 2g-2+n$

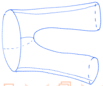
Differential forms $\omega_{g,n}(z_1,\ldots,z_n)$, $z_i \in \Sigma$, $\forall g,n \geq 0$.

- x finitely many simple ramification points ($\mathrm{Cr}(x)$) and y holomorphic around $a \in \mathrm{Cr}(x)$ and $\mathrm{d}y(a) \neq 0 \Rightarrow$ Local involution σ around every ramification point: $x(z) = x(\sigma(z))$.
- $\omega_{0,2}$ symmetric bi-differential on $\Sigma \times \Sigma$ with only double poles along the diagonal and vanishing residues, that is when $z_1 \to z_2$

$$\omega_{0,2}(z_1,z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2} + \overbrace{h(z_1,z_2)}^{\text{holomorphic}}.$$



- Terms in correspondence with the ways of cutting a **pair of pants** $(0,3)$ from $S_{g,n}$.



Symplectic invariance

$$(\Sigma, (x, y)) \overset{\mathsf{TR}}{\rightsquigarrow} \omega_{g,n}(z_1, \ldots, z_n) \; (\omega_{g,0} = \mathfrak{F}_g \in \mathbb{C})$$

Symplectic invariance

$$\begin{tikzcd}
(\Sigma,(x,y)) \arrow[r, rightsquigarrow, "\mathrm{TR}"] \arrow[d, mapsto, "\substack{\Phi \\ \text{preserving} \\ |\mathrm{d}x\wedge\mathrm{d}y|}"'] & \omega_{g,n}(z_1,\ldots,z_n)\ (\omega_{g,0} = \mathfrak{F}_g \in \mathbb{C}) \\
(\Sigma,(\check{x},\check{y})) &
\end{tikzcd}$$

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(\Sigma,(\check{x},\check{y})) \arrow[r, squiggly, "\mathrm{TR}"] & \check{\omega}_{g,n}(z_1,\dots,z_n)\ (\check{\omega}_{g,0} = \check{\mathfrak{F}}_g)
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$\Phi = \mathcal{E}\colon (x,y) \mapsto (y,x)$ not well understood.

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Let $x(z) = \alpha + \gamma(z + \frac{1}{z})$.

Theorem (Eynard, '05)

$$(\mathbb{CP}^1, (x, y = W_1^{[0]}(x)), \omega_{0,2} = B)$$

$$\Big\downarrow \text{TR}$$

$$\frac{\omega_{g,n}(z_1,\dots,z_n)}{dx_1\cdots dx_n} = W_n^{[g]}(x_1,\dots,x_n),$$

$$\forall 2g-2+n>0,\ z_i \to \infty.$$

Maps

$\overset{\mathcal{E}}{\longleftrightarrow}$

Symplectic invariance

$$
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Maps

$\xleftrightarrow{(0,1) \text{ et } (0,2)\ \checkmark}$ $\qquad \xleftrightarrow{\mathcal{E}}$

Theorem (Borot–Charbonnier–G-F, '21)

$$(\mathbb{CP}^1,(y,x),\omega_{0,2}=B)$$

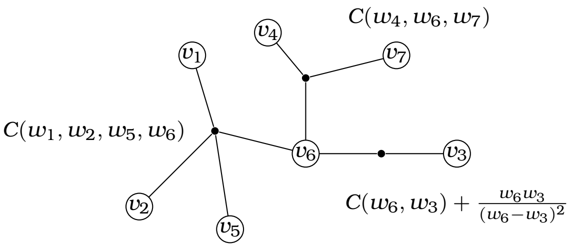
$$\Big\downarrow \text{TR}$$

$$\frac{\check{\omega}_{g,n}(z_1,\ldots,z_n)}{\mathrm{d}y_1\cdots\mathrm{d}y_n} = X_n^{[g]}(y_1,\ldots,y_n),$$

$$\forall 2g-2+n>0,\ z_i\to\infty.$$

Fully simple maps

- Our proof: combinatorial (here x replaces $1/x$ from before).
- Proof by Bychkov–Dunin-Barkowsi–Kazarian–Shadrin, '21: via Fock space formalism.



Thank you for your attention!