Approximation of free convolutions by free infinitely divisible laws

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joint work with:

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Topics

- Optimal approximation by infinitely divisible laws in classical Probability
- Rates of approximation by infinitely divisible laws in free probability
- Lower and upper bounds for classes of measures
- Outline of proofs

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Approximation by Infinitely divisible Laws

Doeblin (1939) constructed i.i.d. random variables $X_1, X_2, \ldots$ s.th.

for any a_n and b_n, any sub sequence n_k of normalized sums

$$b_{n_k}^{-1}(X_1 + \cdots + X_{n_k} - a_{n_k})$$

does not converge to some non degenerate distribution.

Kolmogorov (1953) suggested infinitely divisible distributions for approximating n-fold convolutions $\{\mu^{*n}\}_{n=1}^{\infty}$ for some μ in the distance

$$\Delta(\mu, \nu) = \sup_{x \in \mathbb{R}} |\mu((-\infty, x)) - \nu((-\infty, x))|$$

between p-measures μ and ν. Let $\mathbf{D}^*$: infinitely divisible p-measures.

Prokhorov (1955) proved

$$\Delta(\mu^{*n}, \mathbf{D}^*) := \inf_{\nu \in \mathbf{D}^*} \Delta(\mu^{*n}, \nu) \to 0, \quad n \to \infty, \tag{0.1}$$

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Rate of Approximation by Infinitely divisible Laws

Final answer: Let $\psi(n) := \sup_{\mu \in \mathcal{M}} \Delta(\mu^{*n}, \mathbf{D}^*)$, then

$$c_1 n^{-2/3} \le \psi(n) \le c_2 n^{-2/3},$$

with $c_1, c_2 > 0$ absolute, by Arak (1981,82) (rate n^{-1} for $\hat{\mu} \ge 0$: class $\mathcal{M}_+$)

Chistyakov (1995) extended (0.1) for $\mu \notin \mathbf{D}^*$ via lower bounds of type

$$\Delta(\mu^{*n}, \mathbf{D}^*) \ge \exp\{-b(\mu)\, n\}$$

depending on the moments and the characteristic function (c.f.) of μ.

Moreover, if

$$\limsup_n \frac{\log \Delta(\mu^{*n}, \mathbf{D}^*)}{n} \le -\Delta, \quad \Delta > 0,$$

then the c.f. of μ equals that of an infinitely divisible law on an interval $[-A_\Delta, A_\Delta]$, $\quad A_\Delta > 0$ and such μ exist.

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Rate of Approximation by Infinitely divisible Laws

For $\mu \in \mathcal{M}_+, \mu \notin \mathbf{D}^*$, $c > 0$ absolute.

$$\exp\{-b(\mu)n\} \le \Delta(\mu^{*n}, \mathbf{D}^*) \le c n^{-1}.$$

There exists $\mu \in \mathcal{M}_+$

$$b_1 n^{-k-1} (\log n)^{-2k-3} \le \Delta(\mu^{*n}, \mathbf{D}^*) \le b_2 n^{-k}, \ k \in \mathbf{N}$$

Tools for existence:

undetermined moment problem for $m_k(\mu)$ via complex analysis:

via continuous extension of Hermitian positive functions

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Free Convolutions and Infinite Divisibility

Let $D^{\boxplus}$: infinitely divisible p-measures for free convolution $\boxplus$

$$\Delta(\mu^{\boxplus n}, \mathbf{D}^{\boxplus}) := \inf_{\nu \in \mathbf{D}^{\boxplus}} \Delta(\mu^{\boxplus n}, \nu)$$

Let $\mathbb{C}^{+}$ upper complex plane. For $\mu \in \mathcal{M}$, define Cauchy transform:

$$G_\mu(z) := \int_{-\infty}^{\infty} \frac{\mu(dt)}{z - t}, \quad F_\mu(z) := 1/G_\mu(z), \quad z \in \mathbb{C}^{+},$$

$$c_1(\mu) := \Im(F_\mu(i)) - 1 > 0, \quad \mu \quad \text{non point measure.}$$

Denote by $\mathcal{M}_d$, $d \geq 0$, p-measures with $\beta_d(\mu) := \int_{\mathbb{R}} |x|^d \, \mu(dx) < \infty$.

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Approximation by Free Infinite Divisible Laws

Theorem 1 (Chistyakov-G. (2013))

Let μ denote a p-measure and $c_1(\mu) > 0$. Then

$$\Delta(\mu^{\boxplus n}, \mathbf{D}^{\boxplus}) \le c(\mu)\Big(\int_{[-N_n, N_n]} \frac{|u|}{\sqrt{n}}\,\mu(du) + \mu(\{\mathbb{R} \setminus [-N_n/8, N_n/8]\})\Big), \quad n \in \mathbb{N},$$

where $N_n := \sqrt{c_1(\mu)(n-1)}$. *Hence, analogous to (0.1):*

$$\Delta(\mu^{\boxplus n}, \mathbf{D}^{\boxplus}) \to 0, \quad \text{as} \quad n \to \infty.$$

Moreover, for $\mu \in M_d$, $d > 0$:

$$\Delta(\mu^{\boxplus n}, \mathbf{D}^{\boxplus}) \le c(\mu, d)\, n^{-\min\{d, 1/2\}}, \quad n \in \mathbb{N}, \quad .$$

Convergence holds in variation distance as well.

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$\Delta(\mu^{\boxplus n}, \mathbf{D}^{\boxplus}) \to 0, \quad$ *as* $\quad n \to \infty.$ *Moreover, for* $\mu \in M_d,\ d > 0$ *:*

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Theorem 2 (Chistyakov-G. (2020))

Let $\mu \in \mathcal{M}_d$, $d \in \mathbb{N}, d \geq 3$ *and* $\mu \notin \mathbf{D}^{\boxplus}$. *Then*

$$\liminf_{n\to\infty} \frac{\log \Delta(\mu^{\boxplus n}, \mathbf{D}^{\boxplus})}{\sqrt{n}} > -\infty,$$

hence two-sided bounds:

$$\exp\{-c_2(\mu)\sqrt{n}\} \leq \Delta(\mu^{\boxplus n}, \mathbf{D}^{\boxplus}) \leq \frac{c_3(\mu)}{\sqrt{n}}, \quad n \in \mathbb{N}.$$

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Higher Moments and Polynomial Rates

Let $\mu \in \mathcal{M}_d$ with $d \geq 3$ and $m_1(\mu) = 0, m_2(\mu) = 1$.

Denote $\mu_n((-\infty, x)) := \mu((-\infty, x\sqrt{n}))$, $x \in \mathbb{R}$.

$\boxplus$-infinitely divisible Meixner p-measures $\{w_a : a \in \mathbb{R}\}$

given by its Cauchy transform

$$G_{w_a}(z) = \Big(a + \frac{1}{2}\Big(z - a + \sqrt{(z-a)^2 - 4}\Big)\Big)^{-1}, \quad z \in \mathbb{C}.$$

Theorem 3 ([Chistyakov-G. (2013), PTRF)

For $\mu \in \mathcal{M}_d$ with $d \geq 3$ and absolute constant $c > 0$ and all $n \in \mathbb{N}$

$$\Delta(\mu_n^{\boxplus n}, w_{a_n}) \leq c\frac{\beta_d(\mu)}{n^{(d-2)/2}}, \quad 3 \leq d \leq 4,$$

$$\leq c\frac{\beta_4(\mu)}{n}, \quad d > 4$$

where $a_n := m_3(\mu)/\sqrt{n}$.

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Let $\varphi(t;\mu)$ c.f. of p-measure μ.

Theorem 2 follows from

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Let $\mu \in \mathcal{M}_d$, $d \in \mathbb{N}, d \geq 3$, *and assume*

$$\liminf_{n\to\infty} \frac{\log \Delta(\mu^{\boxplus n}, \mathbf{D}^{\boxplus})}{\sqrt{n}} = -\beta < 0.$$

Then

$$\varphi(t;\mu) = \varphi(t;\rho), \quad t \in [-\frac{\beta}{c_\mu}, \frac{\beta}{c_\mu}], \quad c_\mu := 300\sqrt{m_2(\mu) - m_1^2(\mu)},$$

where ρ *is* $\boxplus$*-infinitely divisible.*

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Let $\mathcal{M}_{det}$: *p*-measures μ uniquely determined by their moment sequence $\{m_k(\mu)\}_{k=0}^{\infty}$ as described in Akhiezer(1965). Then

Corollary 5

Let $\mu \in \mathcal{M}_{det}$ and $\mu \notin \mathbf{D}^{\boxplus}$. Then

$$\liminf_{n\to\infty} \frac{\log \Delta(\mu^{n\boxplus}, \mathbf{D}^{\boxplus})}{\sqrt{n}} = 0. \quad (0.2)$$

$\mathcal{M}_{det}$ contains *p*-measures μ such that Carleman's condition

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{m_{2k}(\mu)}} = \infty.$$

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Slowly varying $L(x) \geq 1$ with continuous derivative and $L(x) \sim L(x/L(x))$ as $x \to \infty$.

Theorem 6

There exists a symmetric p-measure

$$\Delta(\mu^{\boxplus n}, \mathbf{D}^{\boxplus}) \leq e^{-c_4(\mu)\sqrt{n}/L(n)}, \quad n \in \mathbb{N},$$

with $0 < L(n) \geq 1$ *slowly varying positive function with*

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Polynomial Rates and Fixed Cumulants

$\mu \in \mathcal{M}_{2k}$ with free cumulants $\alpha_s(\mu)$, $s = 1, \ldots, 2k$.

Theorem 7

Let $\mu \in \mathcal{M}_{2k}$, $k \in \mathbb{N}, k \geq 4$, *with* $\alpha_{2k}(\mu) < 0$.

Hence $\mu \notin \mathbf{D}^{\boxplus}$. *Assume there exists a* $\boxplus$*-infinitely divisible p-measure* $\rho \in \mathcal{M}_{2k}$ *with*

$$\alpha_s(\mu) = \alpha_s(\rho) \geq 0, \quad s = 1, \ldots, 2k - 1$$

Then there exist $c_{\pm}(\mu, \rho) > 0$ *(depending on* μ *and* ρ *only) s.th.*

$$\frac{c_-(\mu, \rho)}{n^{k+2+\frac{3}{k-1}}} \leq \Delta(\mu^{\boxplus n}, \rho^{\boxplus n}) \leq \frac{c_+(\mu, \rho)}{n^{(k-4)/4}}, \quad n \in \mathbb{N}.$$

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$$\frac{c_-(\mu,\rho)}{n^{k+2+\frac{3}{k-1}}} \leq \Delta(\mu^{\boxplus n}, \rho^{\boxplus n}) \leq \frac{c_+(\mu,\rho)}{n^{(k-4)/4}}, \quad n \in \mathbb{N}.$$

Outline of Proofs

$$G_\mu(z) := \int_{-\infty}^{\infty} \frac{\mu(dt)}{z-t}, \quad F_\mu(z) := 1/G_\mu(z), \quad z \in \mathbb{C}^+,$$

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Show: $Z_n : C_+ \to D_n \subset C_+$ is univalent, mapping **R** to Jordan curve s.th.

$$|Z_n(x)| \geq 1/4\sqrt{c_1(\mu)(n-1)}$$

and $Z_n^{(-1)}$ may be extended from D_n to C_+ via (*).

By Nevanlinna representation there exist τ, $\tau(\mathbf{R}) = c_1(\mu)$:

$$F_\mu(z) = \Re(F_\mu(i)) + z + \int_{\mathbb{R}} \frac{1+uz}{u-z}\,\tau(du), \quad z \in \mathbb{C}^+,$$

which leads via Bercovici and Voiculescu's condition

$$\lim_{y \to +\infty} \frac{\phi_{\rho_n}(y)}{y} = 0.$$

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Idea:

Approximate distribution functions of $\mu^{\boxplus n}$ and ρ_n at x via limits of imaginary parts of

$$G_\mu(Z_n(x+iy)) = \Big(\int_{|u|<c\sqrt{n}} + \int_{|u|>c\sqrt{n}}\Big) \frac{\mu(du)}{Z_n(x+iy)-u}$$

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This leads to the convergence bounds in Theorem 1.

Suffices to show:

Assume $\mu \in \mathcal{M}_d,\ d \geq 3, d \in \mathbf{N}$, and a sequence $\{\nu_n\}_{n=1}^{\infty}$ of p-measures s.th.

$$\liminf_{n\to\infty} \frac{\log \Delta(\mu^{\boxplus n}, \nu_n^{\boxplus n})}{\sqrt{n}} = -\beta < 0.$$

Then a subsequence $\nu_{n'}$ converges weakly to a p-measure ν s.th.

$$\varphi(t;\mu) = \varphi(t;\nu), \quad t \in [-\frac{\beta}{c_\mu}, \frac{\beta}{c_\mu}], \quad c_\mu := 300\sqrt{m_2(\mu) - m_1^2(\mu)}.$$

Hence assume:

$$\Delta(\mu^{\boxplus n}, \nu_n^{\boxplus n}) \leq \varepsilon_n := e^{-\beta'\sqrt{n}},\ \beta' < \beta, \quad m_1(\mu) = 0,$$

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$$|Z_n(z) - W_n(z)| \leq \frac{18\pi\varepsilon_n}{\Im z}\Big(|z| + \frac{n m_2(\mu)}{\Im z}\Big)^2,$$

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Let $N_n := \varepsilon_n^{-1/13}$. Truncate μ and ν_n at $2N_n$ resulting in $\tilde{\mu}, \tilde{\nu_n}$ with corresponding subordinating functions $\tilde{Z}_n(z)$, $\tilde{W}_n(z)$. Show

$$\Delta(\tilde{\mu}^{\boxplus n}, \tilde{\nu}_n^{\boxplus n}) \leq c(\mu)\varepsilon_n^*$$

and relate for $z = x + ih_n$, $h_n \to 0+$ to the size of

$$\int_{[-2nN_n, 2nN_n]} \frac{(\tilde{\mu}^{\boxplus n} - \tilde{\nu_n}^{\boxplus n})(du)}{z - u} = G_{\tilde{\mu}}(\tilde{Z}_n(z)) - G_{\tilde{\rho}_n}(\tilde{W}_n(z))$$

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to show the 2nd integral, say $R_n(z)$, is negligible for for $z = x + Ih_n$.

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using $\quad |\tilde{Z}_n(z) - \tilde{W}_n(z)| \leq c(\mu) n^3 N_n^2 \varepsilon_n^* \quad$ for $\quad 1 \leq \Im z \leq 6nN_n$

to show the 2nd integral, say $R_n(z)$, is negligible for for $z = x + Ih_n$.

Furthermore let $z = \tilde{Z}_n^{(-1)}(x + ih_n)$

$$\int \exp\{itx\}\Im\Big(\int_{[-2nN_n,2nN_n]} \frac{(\tilde{\mu}^{\boxplus n} - \tilde{\nu}_n^{\boxplus n})(du)}{\tilde{Z}_n^{(-1)}(x + ih_n) - u} dx - R_n(\tilde{Z}_n^{(-1)}(x + ih_n))\Big)$$

$$= \int \exp\{itx\}\Im \int_{[-2N_n,2N_n]} \frac{(\tilde{\mu} - \tilde{\rho}_n)(du)}{x + ih_n - u} dx = e^{-h_n|t|}(\varphi(t;\tilde{\mu}) - \varphi(t;\tilde{\rho}_n)),$$

where $\varphi(t;\tilde{\mu})$ and $\varphi(t;\tilde{\rho}_n))$ are the c.f. of $\tilde{\mu}$ and $\tilde{\rho}_n$.

Integrating locally in t yields convergence of c.f. locally.

tightness of $\tilde{\rho}_n$ follows from that of μ

convergence of a subsequence of $\tilde{\rho}_n$ to some p-measure ρ

proves Theorem 2.

Furthermore let $z = \tilde{Z}_n^{(-1)}(x + ih_n)$

$$\int \exp\{itx\} \Im\Big(\int_{[-2nN_n, 2nN_n]} \frac{(\tilde{\mu}^{\boxplus n} - \tilde{\nu}_n^{\boxplus n})(du)}{\tilde{Z}_n^{(-1)}(x + ih_n) - u} dx - R_n(\tilde{Z}_n^{(-1)}(x + ih_n))\Big)$$

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Thank you for your attention!