Sums of Commutators in Free Probability

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https://arxiv.org/abs/2002.06051
https://arxiv.org/abs/2002.06052
https://arxiv.org/abs/2004.02679

The operad of free convolutions

X, Y s.a. free r.v. $\quad X \sim \mu$, $Y \sim \nu$

$p(X,Y)$ s.a. nc-polynomial

$$\mu \boxed{p} \nu := \mu_{p(X,Y)}$$

$p(X,Y) = X + Y \longrightarrow \mu \boxplus \nu$ Voiculescu 1986

$p(X,Y) = X \cdot Y \qquad \mu \boxtimes \nu \qquad$ 1987

$p(X,Y) = i[X,Y] \qquad [\mu \boxminus \nu]$ free commutator

Nica/Speicher 1998

$p(X,Y) = XY + YX$ Vasilchuk 2003

General solution: Linearization

B/M/S 2013

2017

→ numerical approximation

→ no combinatorics

Next step: - higher order polynomials

- more variables

$X + XYX$ Cantelli (L + Szpojankowski 2019)

Vasilchuk 2018

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more variables

sample variance: X_i free i.d.

$$S_n^2 = \sum (X_i - \bar{X})^2$$

$$\bar{X} = \frac{1}{n} \sum X_i$$

The free commutator

Theorem (Nica-Speicher 1998).
The R-transform of the free commutator satisfies the identity

$$R_{[\mu\Box\nu]}(z) = 2(R^E_\mu \boxed{\star} R^E_\nu \boxed{\star} \zeta)(z^2);$$

in particular, ***odd cumulants cancel****.*

$$\begin{aligned} K_2(i[x,y]) = {} & 2\,K_2\,(x,x)\,K_2\,(y,y) - 2\,K_4\,(x,y,x,y) \\ & - 2\,K_2\,(x,y)^2 + 2\,K_4\,(x,x,y,y) \\ K_3(i[x,y]) = {} & 3i\,(K_6\,(x,x,y,x,y,y) - K_6\,(x,x,y,y,x,y)) \end{aligned}$$

$$
\begin{gathered}
K_4(i[x,y]) = 2\,K_2\,(x,x)^2\,K_4\,(y,y,y,y) + 2\,K_4\,(x,x,x,x)\,K_2\,(y,y)^2 \\
+ 2\,K_2\,(x,x)^2\,K_2\,(y,y)^2 - 8\,K_2\,(x,x)\,K_4\,(x,y,x,y)\,K_2\,(y,y) \\
- 4\,K_2\,(x,x)\,K_2\,(x,y)^2\,K_2\,(y,y) - 8\,K_4\,(x,x,x,y)\,K_2\,(x,y)\,K_2\,(y,y) \\
+ 12\,K_2\,(x,x)\,K_4\,(x,x,y,y)\,K_2\,(y,y) + 4\,K_6\,(x,x,y,x,x,y)\,K_2\,(y,y) \\
- 8\,K_6\,(x,x,x,y,x,y)\,K_2\,(y,y) + 4\,K_6\,(x,x,x,x,y,y)\,K_2\,(y,y) \\
- 8\,K_2\,(x,x)\,K_2\,(x,y)\,K_4\,(x,y,y,y) + 4\,K_2\,(x,x)\,K_6\,(x,y,y,x,y,y) \\
- 8\,K_2\,(x,x)\,K_6\,(x,y,x,y,y,y) + 2\,K_8\,(x,y,x,y,x,y,x,y) \\
+ 8\,K_2\,(x,y)\,K_6\,(x,y,x,y,x,y) + 4\,K_4\,(x,y,x,y)^2 \\
12\,K_2\,(x,y)^2\,K_4\,(x,y,x,y) - 8\,K_4\,(x,x,y,y)\,K_4\,(x,y,x,y) \\
+ 2\,K_2\,(x,y)^4 - 4\,K_4\,(x,x,y,y)\,K_2\,(x,y)^2 \\
- 4\,K_6\,(x,x,y,y,x,y)\,K_2\,(x,y) - 4\,K_6\,(x,x,y,x,y,y)\,K_2\,(x,y) \\
+ 4\,K_2\,(x,x)\,K_6\,(x,x,y,y,y,y) - 4\,K_8\,(x,x,y,y,x,y,x,y) \\
+ 2\,K_8\,(x,x,y,y,x,x,y,y) + 4\,K_4\,(x,x,y,y)^2 \\
+ 4\,K_8\,(x,x,y,x,y,y,x,y) - 4\,K_8\,(x,x,y,x,y,x,y,y)
\end{gathered}
$$

Theorem (Arizmendi-Hasebe-Sakuma 2013).

1. *If μ and ν are freely infinitely divisible, then so is $[\mu \,\square\, \nu]$.*

2. *If μ is even and FID, then μ^2 is FID.*

Question
Are there other polynomials which preserve free infinite divisibility?

μ^2: if $X \sim \mu$ then $\mu^2 := \mu_{X^2}$

Free i.d. families

Lemma (E-L 2017).
The sample variance cancels odd cumulants.

X_i free i.d.

$$S_n^2 = \sum (X_i - \bar{X})^2$$

Rubin's problem:

assume X_i i.i.d

s.t. $S_n^2 \sim \chi^2$-distribution

$\overset{?}{\Longrightarrow}$ X_i are normal free analog: X_i free i.d.

s.v. $S_n^2 \sim MP$

$\overset{?}{\Rightarrow}$ X_i are semicircular

<u>no</u> : only even cumulants vanish

$\exists$ distrib. with fin. many cum

e.g. $\kappa_1 = 0$, $\kappa_2 = 1$, $\kappa_3 = \varepsilon$, $\kappa_n = 0 \ \forall n \geq 4$

Lemma (E-L 2017).
Assume

1. $X_1, X_2, \dots, X_n$ *free i.d. copies of* X

2. $L = \sum_{i=1}^n \alpha_i X_i$ *a linear form with* $\sum \alpha_i = 0$

Then the quadratic form

$$P = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} L_\sigma^2,$$

cancels odd cumulants.

$$L = X_1 - \frac{1}{n} \sum_{i=1}^{n} X_i$$

other examples: X_1, X_2, X_3 free i.d.

$$(X_1 - X_2)^2 + i[X_1, X_2]$$ cancels odd cum

$$(X_1 - X_2)^2 + i[X_2, X_3]$$ does not cancel oddam.

Proposition (E-L 2017).
Assume

1. $X_1, X_2, \ldots, X_n$ *free family of s.a. even FID random variables.*
2. $A = [a_{i,j}]_{i,j=1}^n \in M_n(\mathbb{C})$ *a selfadjoint matrix,*

Then the distribution of $Q = \sum_{i,j}^n a_{i,j} X_i X_j$ *is FID.*

any quadr. fn in FID variables
even
is FID

Corollary (E-L 2017).
Assume that the s.a. quadratic form

$$Q = \sum a_{ij} X_i X_j$$

cancels odd cumulants for all free/free i.d. families.
Then Q preserves FID for all free/free i.d. families.

In particular, the sample variance preserves FID for free i.d. families.

Q: if quadratic form preserves FID for any iid sequence $\overset{?}{\Longrightarrow}$ cancels odd cumulants?

$[[X_1, X_2], X_3]$

Theorem.
The following properties are equivalent for a quadratic form

$$Q_n = \sum_{i,j=1}^{n} a_{i,j} X_i X_j$$

with selfadjoint system matrix $A = [a_{i,j}]_{i,j=1}^{n} \in M_n(\mathbb{C})$.

(i) Q_n *cancels odd cumulants for any family* $X_1, X_2, \ldots, X_n$.

(ii) Q_n *cancels odd cumulants for any free family* $X_1, X_2, \ldots, X_n$.

(iii) Q_n *preserves free infinite divisibility for free families.*

(iv) A *is skew symmetric or equivalently,*

$Q_n = \sum_{k<l} a_{k,l}(X_k X_l - X_l X_k)$ *is a sum of commutators.*

(v) *The distribution of* Q_n *is symmetric for any free family* $X_1, X_2, \ldots, X_n$.

$A^* = A \qquad A^T = -A$

Higher order polynomials

- Free and strong cancellation of odd cumulants not equivalent:

$$K_3([[X_1, X_2], X_1]) = -6r_2r_3r_4 + 6r_3^3 - 6r_2^3r_3$$

- Assume X, Y FID, Z arbitrary, then $[X, Y]Z[X, Y]$ is FID but does not cancel odd cumulants. ↪ comp. Poisson

- Open Question:

  Cancellation of odd cumulants $\Longrightarrow$ preservation of FID?

$$[[X_1, X_2], X_3]$$

cancel odd cu if X_1, X_2, X_3 are free

Ingredients of the proof

Product formula (James 1958/Leonov-Shiryaev 1959/Krawczyk-Speicher 2000)

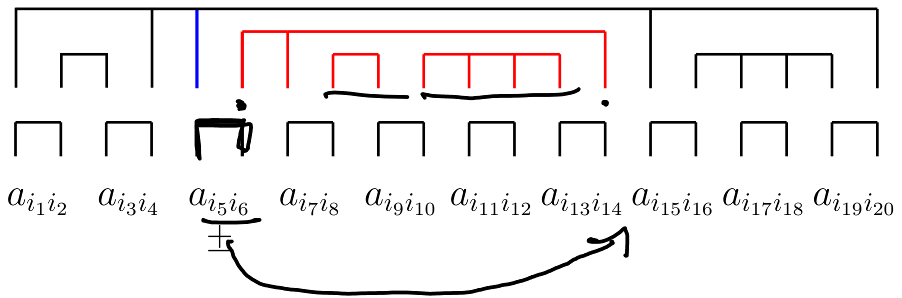
$$K_r(X_1X_2, X_3X_4, \dots, X_{2r-1}X_{2r}) = \sum_{\substack{\pi \in NC(\cancel{n}\, 2r) \\ \pi \vee \sqcap\,\sqcap \cdots \sqcap = \sqcap \cdots \sqcap}} K_\pi(X_1, X_2, \dots, X_n).$$

$$Q = \sum_{k<l} a_{kl}\, i[X_k, X_l] \qquad \pi \in NC(2r)$$

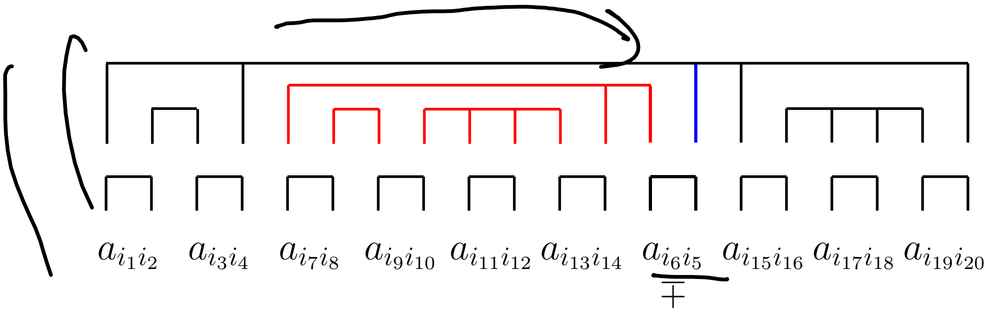
$$K_r(Q) = \sum_{i_1 i_2 \dots i_{2r}} a_{i_1 i_2} a_{i_3 i_4} \cdots a_{i_{2r-1} i_{2r}} \Big(K_r(X_{i_1} X_{i_2}, \dots, X_{i_{2r-1}} X_{i_{2r}}) \Big)$$

idea: in this sum, the odd cumulants vanish

pick the leftmost inner odd block



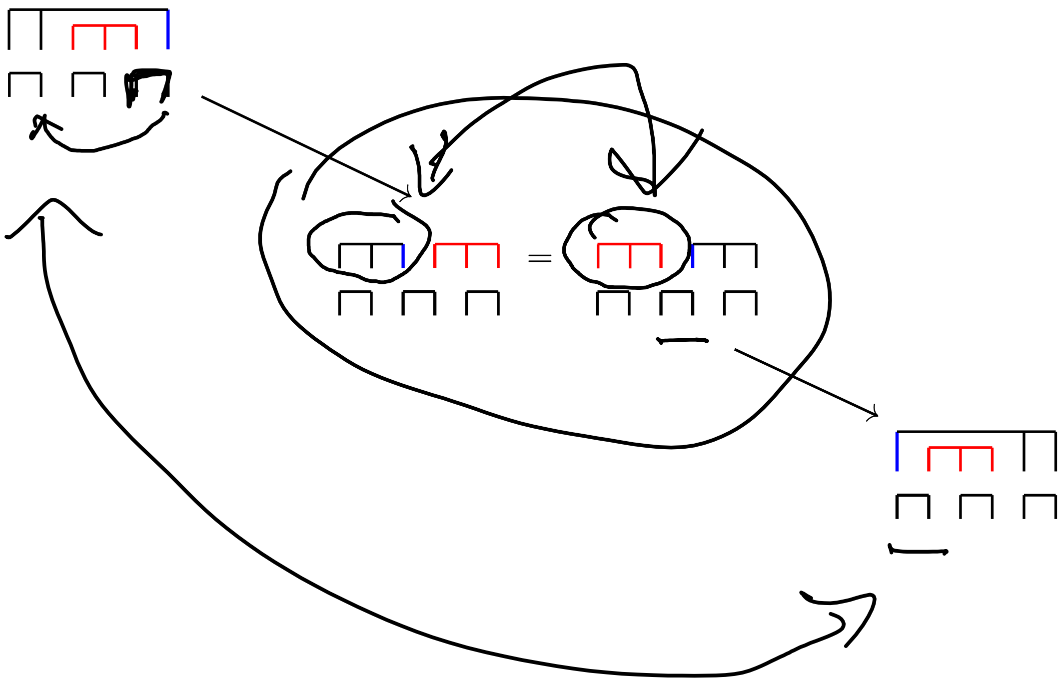
$K_3(X_6, X_{13}, X_{14})$



trace $\Rightarrow$ $K_3(X_{13}, X_{14}, X_6)$

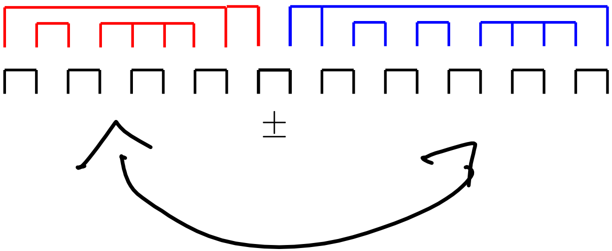
→ not an involution

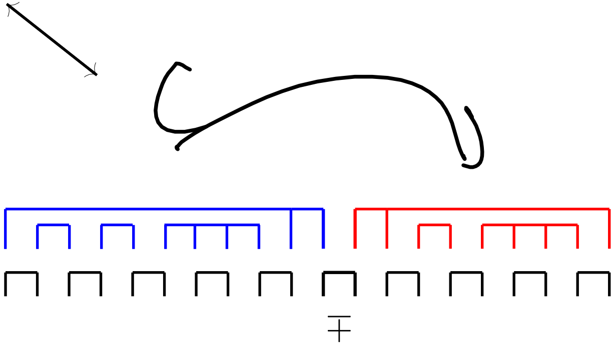
Problem:



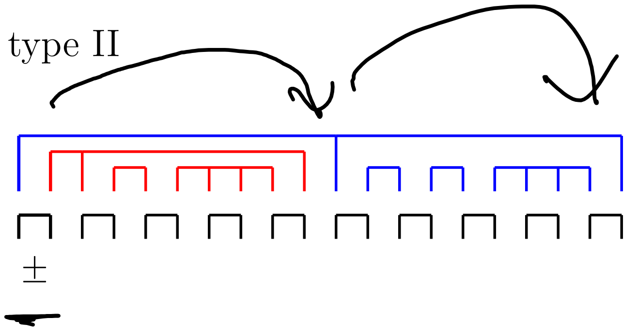
Solution: Extra treatment of certain partitions.

type I

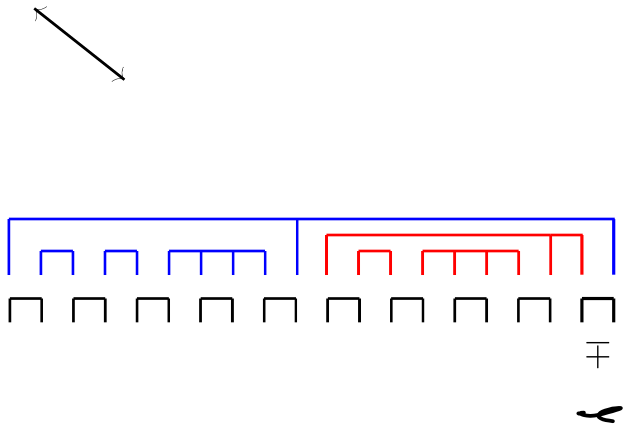




type II



$\rightarrow$ is an involution



Limit theorem for quadratic forms

Theorem.
$A_n = [a_{i,j}^{(n)}] \in M_n(\mathbb{C})$ s.a. matrices s.t.

1. $\sup_{i,j,n} \left| a_{i,j}^{(n)} \right| < \infty$ not a proba m
2. $\frac{1}{n} A_n \xrightarrow{d} \mu$ *w.r.t. to the nonnormalized trace.*

X_i be free copies of a centered r.v. X of variance 1
Then the sequence of quadratic forms

$$Q_n = \frac{1}{n} \sum_{i,j=1}^{n} a_{i,j}^{(n)} X_i X_j$$

converges in distribution to Y, where

$$K_r(Y) = \int t^r \, d\mu(t).$$

Proof $K_r(Q_n) =$

$$= \frac{1}{n^r} \sum_{i_1 \dots i_{2r}} a_{i_1 i_2} \cdots a_{i_{2r-1} i_{2r}} \sum_{\pi \vee \sqcap \dots \sqcap = 1_{2r}} K_\pi(x_{i_1}, \dots, x_{i_{2r}})$$

$\tau(X_i) = 0 \Rightarrow$ no singletons

$\Rightarrow \leq r$ blocks

only π with r blocks contribute

$\Rightarrow \pi$ is pair partition

$\Rightarrow$ wlog assume X_i are semicircular

Quadratic forms in semicirculars

Let $S_1, S_2, \ldots, S_n$ be a standard semicircular family, then

$$K_r\left(\sum a_{ij} S_i S_j\right) = \mathrm{Tr}(A^r)$$

→ diagonalize the matrix

$\sim \sum \lambda_i Y_i \qquad Y_i \sim MP$

→ $\sum$

π pair partition

→ only partition:

$$K_r\left(\sum a_{ij} S_i S_j\right) = \sum_{i_1, \ldots, i_r} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_r i_1} \cdot 1$$

Generalized Tetilla laws

$$Q_n = \sum_{k<l} i[X_k, X_l] \qquad A_n = \begin{bmatrix} 0 & i & \dots & i \\ -i & 0 & \dots & i \\ \dots & \dots & \dots & \dots \\ -i & -i & \dots & 0 \end{bmatrix}$$

has eigenvalues

$$\lambda_k = \cot\left(\frac{k+\frac{1}{2}}{n}\right)\pi, \qquad k = 0, 1, \dots, n-1$$

and thus

$$K_{2m}(\sum_{k<l} i[S_k, S_l]) = \mathrm{Tr}(A^{2m}) = \sum_{k=0}^{n-1} \cot^{2m}\left(\frac{k+\frac{1}{2}}{n}\right)\pi$$

And now for something completely different

Lemma. *Let* $a = \cot\alpha$*, then the eigenvalues of the matrix*

$$C_n = \begin{bmatrix} a & a+i & \dots & a+i \\ a-i & a & \dots & a+i \\ \dots & \dots & \dots & \dots \\ a-i & a-i & \dots & a \end{bmatrix} \in M_n(\mathbb{C})$$

are

$$\lambda_k = \cot\left(\frac{\alpha + k\pi}{n}\right) \qquad k = 0, 1, \dots, n-1$$

Corollary. *The sum*

$$S(m,n,\alpha) = \sum_{k=0}^{n-1} \cot^m\left(\frac{\alpha+k\pi}{n}\right) = \mathrm{Tr}\left(C_n^{2m}\right)$$

is a polynomial in $\cot\alpha$ *with positive integer coefficients.*

More precisely,

$$\boxed{S(m,n,\alpha) = (-1)^{m/2} n\,\mathbb{1}_{m \text{ even}} + \frac{1}{(m-1)!}\sum_{k=1}^{m} n^k \underline{A_m^{(k)} P_{k-1}(\cot\alpha)}}$$

where $P_k(x)$ are the **derivative polynomials** of $\tan\alpha$:

$$\frac{d^k}{d\alpha^k}\tan\alpha = P_k(\tan\alpha) \qquad \frac{d^k}{d\alpha^k}\cot\alpha = (-1)^k P_k(\cot\alpha)$$

$$\frac{d}{d\alpha}\tan\alpha = 1 + \tan^2\alpha \qquad P_1(x) = 1 + x^2$$

$$\frac{d^n}{d\alpha^n}\tan\alpha = \frac{d}{d\alpha} P_{n-1}(\tan\alpha) = P'_{n-1}(\tan\alpha)(1+\tan^2\alpha)$$

$$P_n(x) = (1+x^2)P'_{n-1}(x)$$

identity:

$$x^m = \frac{1}{(m-1)!}\sum_{k=1}^{m} A_m^{(k)} P_{k-1}(x) + (-1)^{m/2}[m \text{ even}]$$

where $A_m^{(k)}$ are the **arctangent numbers**

$$\frac{(\arctan z)^k}{k!} = \sum_{n=k}^{\infty} \frac{A_n^{(k)}}{n!} z^n;$$

$$\sum_s \cot^m\left(\frac{\alpha+s\pi}{n}\right) = \sum \frac{1}{(m-1)!} \sum A_m^{(k)} P_{k-1}\left(\cot\frac{\alpha+s\pi}{n}\right)$$

$$(-1)^{n-1} n^{k-1} \frac{d^{n-1}}{d\alpha^{n-1}} \overline{\cot\left(\frac{\alpha+s\pi}{n}\right)}$$

$$\frac{d^{n-1}}{d\alpha^{n-1}} \sum_{s=0}^{n-1} \cot\left(\frac{\alpha+s\pi}{n}\right)$$

$$(A_n) = n \cdot \cot$$

Limit law of sum of commutators

$$K_{2m}\Big(\frac{1}{n}\sum_{k<l} i[S_k,S_l]\Big) = \frac{1}{n^{2m}}\left((-1)^m n + \frac{1}{(2m-1)!}\sum_{k=1}^{2m} n^k A_{2m}^{(k)}\, P_{k-1}(\cot(\pi/2))\right)$$

0

1

Lead:

$$\frac{1}{(2m-1)!} A_{2m}^{(2m)} P_{2m-1}(0) \;\; 1 \;\; + O\left(\frac{1}{n}\right)$$

$$\sum_{k=0}^{\infty} \frac{P_k(0)}{k!} z^k = \sum_{k=0}^{\infty} \frac{\tan^{(k)}(0)}{k!} z^k = \sum_{k=0}^{\infty} \frac{T_k}{k!} z^k$$

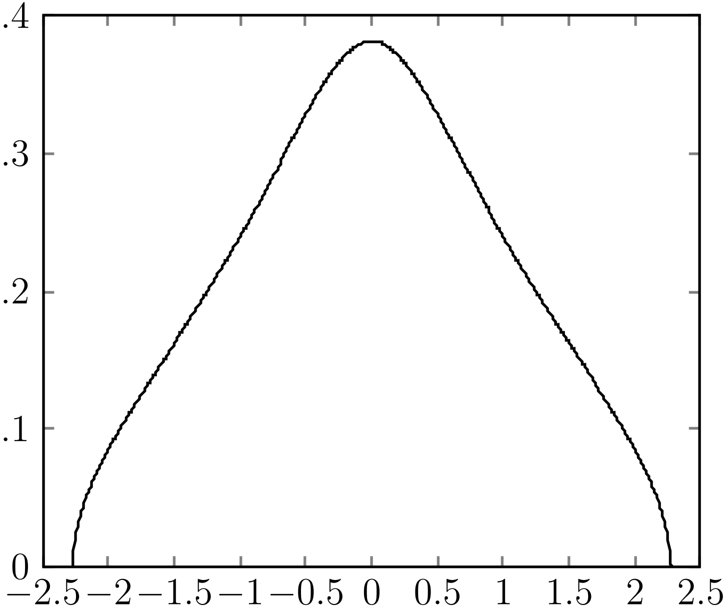
$= \tan z$

tangent numbers

the limit law is the *free tangent law* with R-transform

$$R_T(z) = \tan z = \sum \frac{T_n}{n!} z^n$$

where $T_{2n-1} = (-1)^{k+1} \frac{4^n(4^n-1)B_{2n}}{2n}$ are the *tangent numbers*.



Density of the free tangent law

$$\rho = \inf_{t>0} \frac{1}{t} + \tan t = \frac{1}{\omega}\left(1+\sqrt{1-\omega^2}\right)$$

ω = Arakelian number

$$\cos x = x$$

$$\rho \approx 2.264437$$

Arakelian

sum of commutators and anticommutators

$$\cancel{a}\sum\{X_k, X_l\} + \cancel{b}i\sum[X_k, X_l] \qquad A_n = \begin{bmatrix} 0 & a+bi & \dots & a+bi \\ a-bi & 0 & \dots & a+bi \\ \dots & \dots & \dots & \dots \\ a-ib & a-bi & \dots & 0 \end{bmatrix}$$

the limit law has R-transform

$$R_Y(z) = \frac{\tan(bz)}{b - a\tan(bz)}$$

for $a = b = 1$ this is the *free zigzag law* with R-transform

$$R(z) = \tan z + \sec z = \sum \frac{E_n}{n!} z^n$$

where E_n are Euler's *zigzag numbers.*

$E_n = \#$ zigzag perms