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Free Orthogonal Quantum Groups and Strong 1-Boundedness

UC Berkeley Probabilistic Operator Algebra Seminar

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Main Result

Main Question: can we distinguish the free group factors and the unimodular free orthogonal quantum group factors?
Answer: yes!
How: through strong 1-boundedness.

Theorem (Brannan–Vergnioux '18, E. '20)

Let $Q \in \mathrm{GL}_N(\mathbb{C})$, $N \geq 3$, *be such that* $Q\overline{Q} \in \mathbb{C}I_N$ *and such that* $\mathbb{F}O(Q)$ *is unimodular. That is,* $Q \sim I_N$ *(B.–V.) or* $Q \sim J_N$ *(E.). Then* $\mathcal{L}\mathbb{F}O(Q)$ *is a strongly 1-bounded von Neumann algebra and hence not isomorphic to a(n interpolated) free group factor* $\mathcal{L}\mathbb{F}_m$ *with* $m \geq 2$.

Introduction

This talk is based on arXiv: 2006.13648 (*Strong 1-Boundedness of Unimodular Free Orthogonal Quantum Groups*).

Overview of the talk:

- free orthogonal quantum groups,
- strong 1-boundedness,
- sufficient conditions for 1-boundedness,
- upgrading to strong 1-boundedness,
- application to free orthogonal quantum groups,
- and quantum Cayley graphs.

Compact Matrix Quantum Groups

Definition (Woronowicz)

$\mathcal{A}$ unital C*-algebra generated by u_{ij}, $1 \leq i, j \leq N$, s.t. $u = (u_{ij})_{ij}$ (the fundamental corepresentation) and $\overline{u} = (u_{ij}^*)_{ij}$ are invertible in $M_N(\mathbb{C}) \otimes \mathcal{A}$. Let $\Delta \colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be a unital $*$-homomorphism s.t.

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj} \qquad \left(\text{c.f. } (\pi(gh))_{ij} = \sum_k (\pi(g))_{ik}(\pi(h))_{kj}\right).$$

Then the pair $(\mathcal{A}, \Delta)$ is called a compact matrix quantum group.

Consequences:

- Δ is coassociative and hence a comultiplication.
- Unique invariant Haar state h: $(h \otimes \iota)\Delta(\cdot) = h(\cdot)1 = (\iota \otimes h)\Delta(\cdot)$.
- $*$-Algebra level: anti-automorphism S s.t. $(\iota \otimes S)(u) = u^{-1}$, called the antipode.

Free Orthogonal Quantum Groups

Definition (van Daele–Wang)

Let $N \geq 2$ and $Q \in \mathrm{GL}_N(\mathbb{C})$ such that $Q\overline{Q} \in \mathbb{C}I_N$. The free orthogonal quantum group $\mathbb{F}O(Q)$ is the compact matrix quantum group given by the C*-algebra

$$C^*\mathbb{F}O(Q) = \left\langle u_{ij} \,\middle|\, 1 \leq i,j \leq N,\ u \text{ unitary},\ Q\overline{u}Q^{-1} = u \right\rangle.$$

- Note that $\mathbb{F}O(Q)$ is a *virtual object*.
- The case $\mathbb{F}O_M$ ($Q = I_M$) was studied first. It is a *liberation* of $C(O_M)$ often denoted by $C^u(O_M^+)$ or $A_o(M)$.
- Setting $u_{ij} = 0$ for $i \neq j$ gives a surjection onto $C^*(\mathbb{Z}_2)^{\star M}$ (*matricial analogue*), hence the notation $C^*\mathbb{F}O_M$.
- We will view $\mathbb{F}O(Q)$ as a *discrete* quantum group.
- Pontryagin duality applies to all (*locally compact*) quantum groups, and interchanges *compact* and *discrete* quantum groups.

Free Orthogonal Quantum Groups II

Definition

We say that $\mathbb{F}O(Q)$ is unimodular if h is a trace.

- In the unimodular case: $(\iota \otimes S)(u) = u^*$, and S is an involution defined on all of $C^*\mathbb{F}O(Q)$.
- It turns out that unimodularity only happens for $Q = I_M$ or $Q = J_{2N}$ (up to isomorphism), with J_{2N} standard symplectic.
- Write $\mathbb{F}O_{2N}^J = \mathbb{F}O(J_{2N})$. This is a *twist* of $\mathbb{F}O_{2N}$.
- Write $\ell^2\mathbb{F}O(Q)$ for the GNS representation with respect to h, and ξ_0 for the canonical cyclic vector.
- This gives rise to a reduced quantum group C*-algebra $C_{\mathrm{r}}^*\mathbb{F}O(Q)$, and a quantum group von Neumann algebra $\mathcal{L}\mathbb{F}O(Q)$.

Free Orthogonal Quantum Groups III

- S gives an involutive unitary $U(x.\xi_0) = S(x).\xi_0$ on $\ell^2\mathbb{F}O(Q)$.
- One can define a unitary V on $\ell^2\mathbb{F}O(Q)^{\otimes 2}$, such that

$$\Delta(x).(\xi_0 \otimes \xi_0) = V(x \otimes 1).V^*(\xi_0 \otimes \xi_0) \text{ for all } x \in C^*\mathbb{F}O(Q).$$

- This is called the multiplicative unitary of $\mathbb{F}O(Q)$.

Definition

A unitary corepresentation of $\mathbb{F}O(Q)$ on a Hilbert space $\mathcal{H}$ is a unitary operator v in the multiplier algebra $M(\mathcal{K}(\mathcal{H}) \otimes C^*\mathbb{F}O(Q))$ s.t. $(\iota \otimes \Delta)v = v_{12}v_{13} \in M(\mathcal{K}(\mathcal{H}) \otimes C^*\mathbb{F}O(Q)^{\otimes 2})$.

- Important examples: $(v_{\text{triv}}, \mathbb{C})$, $(V, \ell^2\mathbb{F}O(Q))$, and $(u, \mathbb{C}^N)$.
- Finite dimensional unitary corepresentations form a rigid C*-tensor category (Woronowicz–Tannaka–Krein duality).

Free Orthogonal Quantum Groups IV

Definition

If U and W are finite dimensional corepresentations, then an operator $T\colon \mathcal{H}_U \to \mathcal{H}_W$ intertwines U and W if $(T \otimes 1)U = W(T \otimes 1)$. $\mathrm{End}(U)$ is a C*-algebra, and if it is isomorphic to $\mathbb{C}$, we say that U is irreducible.

Theorem (Peter–Weyl for quantum groups)

Every unitary corepresentation decomposes into a direct sum of finite dimensional irreducible unitary corepresentations.

Corollary

Let $\mathrm{Irr}(Q)$ *be a set of representatives of the irreducible corepresentations of* $\mathbb{F}O(Q)$ *containing* u_{triv} *and* u*. Then* $\oplus_{v\in\mathrm{Irr}(Q)} B(\mathcal{H}_v)$ *is dense in* $\ell^2\mathbb{F}O(Q)$*. On this subspace,* $V = \sum_v v$ *acting by left multiplication. We also obtain orthogonal projections* $p_{0,1}$ *such that* $p_0\ell^2\mathbb{F}O(Q) = \mathbb{C}\xi_0$*, and* $p_1\ell^2\mathbb{F}O(Q) = B(\mathcal{H}_u) \cong M_N(\mathbb{C})$*.*

Group-Like Properties

Several approximation properties of discrete groups are captured by their associated von Neumann algebras.

- Examples: amenability, the Haagerup property, weak amenability, and Kazhdan's property (T).

Thus there are meaningful analogues for quantum groups.

- For $\mathbb{F}O(Q)$:
  - if $Q \in \mathrm{GL}_N(\mathbb{C})$ with $N \geq 3$, $\mathbb{F}O(Q)$ is not amenable (Banica);
  - $\mathbb{F}O(Q)$ has the Haagerup property and is weakly amenable with Cowling-Haagerup constant 1 (De Commer–Freslon–Yamashita).
- The free groups $\mathbb{F}_m$ are also non-amenable and weakly amenable with Cowling–Haagerup constant 1, and they have the Haagerup property.
- The analogy persists on the von Neumann algebra level.

von Neumann Algebraic Properties

The FGFs $\mathcal{L}\mathbb{F}_m$ and the unimodular free orthogonal quantum group von Neumann algebras $\mathcal{L}\mathbb{F}O_M$ and $\mathcal{L}\mathbb{F}O_{2N}^J$ share many von Neumann algebraic properties.

Examples:

- type II_1-factors (Vaes–Vergnioux),
- strongly solid and in particular possess no Cartan subalgebra (Caspers, Fima–Vergnioux, Isono),
- full and hence prime (Vaes–Vergnioux),
- $\{\mathcal{L}\mathbb{F}O_M\}$ has FGF-like asymptotics in a strong sense (Banica, Brannan).

Nevertheless, isomorphisms were not expected, as the first L^2-Betti numbers of $\mathbb{F}O_M$ and $\mathbb{F}O_{2N}^J$ vanish (Vernioux, Bichon), while those of $\mathbb{F}_m$ do not.

Strong 1-Boundedness

Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal tracial state τ, and let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$ be self-adjoint elements of $\mathcal{M}$.

- Recall the relative microstates free entropy $\chi(X : Y)$ and the non-microstates free entropy $\chi^*(X)$.
- These give rise to the (modified) microstates free entropy dimension

$$\delta_0(X) = n + \limsup_{\varepsilon \downarrow 0} \frac{\chi(X + \varepsilon S : S)}{|\log \varepsilon|} \leq n,$$

  and the non-microstates free entropy dimension δ^* by the same formula with $\chi^*(X + \varepsilon S)$.
- Here S is an n-tuple of free semicircular elements, free from X.
- Unknown if δ_0 is an invariant. If it is, that will settle the free group factor isomorphism problem.

Strong 1-Boundedness II

Definition (Jung)

X is 1-bounded (for δ_0) if for small ε we have the estimate

$$\chi(X + \varepsilon S : S) \leq (1 - n)|\log \varepsilon| + \text{const}.$$

If additionally an X_j satisfies $\chi(X_j) > -\infty$, X is strongly 1-bounded. $\mathcal{M}$ is called strongly 1-bounded if it admits such a generating tuple.

- One can analogously define 1-boundedness for δ^*, which implies 1-boundedness for δ_0 by $\chi \leq \chi^*$.
- 1-Boundedness is slightly stronger than $\delta_0(X) \leq 1$.

Theorem (Jung)

If $\mathcal{M}$ is strongly 1-bounded, then all generating tuples Y have $\delta_0(Y) \leq 1$.

- $\mathcal{L}\mathbb{F}_m$ admits a generating set Y with $\delta_0(Y) = m$.

Sufficient Condition for 1-Boundedness

- $T = T_1, \ldots, T_n$ noncommutative self-adjoint formal indeterminates, $\mathcal{C} = \mathbb{C}\langle T \rangle$ the $*$-algebra they generate.
- Recall the partial free derivatives $\partial_i \colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ defined by $\partial_i T_j = \delta_{ij}(1 \otimes 1)$ and $\partial(P_1 P_2) = (\partial_i P_1)(1 \otimes P_2) + (P_1 \otimes 1)(\partial_i P_2)$.
- For $P = (P_1, \ldots, P_m) \in \mathcal{C}^m$, define $\partial P \in M_{m \times n}(\mathbb{C}) \otimes \mathcal{C} \otimes \mathcal{C}$ by $(\partial P)_{ij} = \partial_i P_j$.
- These expressions can be evaluated in X, and then $\partial P(X)$ becomes an $\mathcal{M} \otimes \mathcal{M}^{\mathrm{op}}$-module map.
- Hence the image of $\partial P(X)$ can be assigned a Lück–von Neumann dimension, called the rank of ∂P.
- Will also need the Fuglede–Kadison–Lück determinant defined for arbitrary $x \in \mathcal{M}$ by

$$\det_{\mathrm{FKL}}(x) = \exp\big(\tau[\log_+(|x|)]\big).$$

- Here, $\log_+(t) = \log(t)$ for $t > 0$ and 0 else.

Sufficient Condition for 1-Boundedness II

Theorem (Jung, Shlyakhtenko)

Let $\mathcal{M}$ be a von Neumann algebra with faithful normal tracial state τ, and $X = (X_1, \ldots, X_n)$ self-adjoint in $\mathcal{M}$. Assume that there is a vector $F \in \mathcal{C}^m$ such that

$$F(X) = 0 \text{ and } \det_{\mathrm{FKL}} \left[\partial F(X)^* \partial F(X)\right] \neq 0.$$

Then it holds that X is α-bounded (for both δ_0 and δ^*) with

$$\alpha = n - \operatorname{rank} \partial F(X).$$

- Idea: Let F be the defining relations of $\mathbb{F}O(Q)$.
- Fact: $\operatorname{rank} \partial F(u) = M^2 - 1$ for $Q = I_M$ and $= (2N)^2 - 1$ for $Q = J_{2N}$.

Sketch of Shlyakhtenko's Proof

- Recall that $-\chi^*(X)$ is defined through the free Fisher information: $\Phi^*(X) = \sum_i \|\xi_i\|_2^2$ if conjugate variables exist and $+\infty$ otherwise.
- Let $\mathbb{E}_\varepsilon$ be the condition expectation from $W^*(X,S)$ to $W^*(X+\sqrt{\varepsilon}S)$, set $\xi_\varepsilon = \varepsilon^{-1/2}\mathbb{E}_\varepsilon(S)$, then $\Phi^*(X+\sqrt{\varepsilon}S) = \|\xi_\varepsilon\|_2^2$.
- Want to construct a suitable projection to get a lower bound.
- Write E' for orthogonal projection onto the closure of $\mathrm{span}(W^*(X)SW^*(X)) \subset L^2(W^*(X,S))$, write $D = \partial F(X)^* \partial F(X)$, and write $P_\lambda = \chi_{[\lambda,\infty)}(D)$.
- $F(X) = 0$ eventually gives $D\#E'(\xi_\varepsilon) = \varepsilon^{-1/2} D\#S + \mathcal{O}(1)$ for small ε.

$$\|\xi_\varepsilon\|_2^2 \geq \left\|P_{\varepsilon^{1/4}} E'(\xi_\varepsilon)\right\|_2^2 \geq \frac{\tau(P_{\varepsilon^{1/4}})}{\varepsilon} - \frac{K}{\varepsilon^{3/4}} = \frac{\mathrm{rank}\ \partial F(X)}{\varepsilon} - \frac{\phi(\varepsilon^{1/4})}{\varepsilon} + f(\varepsilon)$$

- Here, f is integrable for small ε. D being determinant class gives integrability of $\phi(\varepsilon^{1/4})/\varepsilon$.

Computing rank $\partial F(u)$

Lemma (Shlyakhtenko, Brannan–Vergnioux)

For both $Q = I_N, J_N$, rank $\partial F(u) = N^2 - 1$.

- F consisting of the defining relations gives module isomorphism

$$\mathrm{Der}(\mathbb{C}[\mathbb{F}O(Q)]; \ell^2\mathbb{F}O(Q) \otimes \ell^2\mathbb{F}O(Q)^{\mathrm{op}}) \cong \mathrm{Ker}\ \partial F(u).$$

- Therefore one has that

$$\mathrm{rk}\ \partial F(u) = N^2 - \dim_{\mathcal{L}\mathbb{F}O(Q) \otimes \mathcal{L}\mathbb{F}O(Q)^{\mathrm{op}}} \mathrm{Der}(\mathbb{C}[\mathbb{F}O(Q)]; \ell^2\mathbb{F}O(Q) \otimes \ell^2\mathbb{F}O(Q)^{\mathrm{op}}).$$

- Exact sequences from Hochschild cohomology give

$$\mathrm{rank}\ \partial F(u) = N^2 - \left(\beta_1^{(2)}(\mathbb{F}O(Q)) - \beta_0^{(2)}(\mathbb{F}O(Q)) + 1\right).$$

- The Lemma now follows from the vanishing of the L^2-Betti numbers (Vergnioux, Bichon).

Upgrading to Strong 1-Boundedness

- We now have a strategy to obtain 1-boundedness.
- Fact: $\chi(u_{ij}) > -\infty$ for any $1 \leq i, j \leq M$ when $Q = I_M$ (Banica–Collins–Zinn-Justin).
- **Obstacle:** This is not known for $Q = J_{2N}$.
- Idea: the fundamental character $\psi^u = (\mathrm{Tr} \otimes \iota)(u)$ is semicircular for all Q (Banica), hence has finite microstates free entropy.
- This is a linear combination of generators, and so (u, ψ^u) also generates $\mathcal{L}\mathbb{F}O(Q)$.
- Can we add elements to a 1-bounded set without spoiling 1-boundedness?
- Voiculescu III: conditions under which $\delta_0(X, Y) \leq \delta_0(X)$.

Upgrading to Strong 1-Boundedness II

Lemma

X and Y self-adjoint tuples $(\mathcal{M}, \tau)$ such that $Y \in W^*(X)$ (redundancy). S a free standard semicircular family, free from X, and assume that

$$\varepsilon^{-1} d_2(Y_j; X)(\varepsilon) = \varepsilon^{-1} \inf \left\{ \| Y_j - T \|_2 \,\middle|\, T \in W^*(X + \varepsilon S) \right\}.$$

is bounded around $\varepsilon = 0$ for all $1 \leq j \leq m$ (regularity). Then if X is α-bounded, so is (X, Y).

- Goal:
  $\chi(X + \varepsilon S, Y_1 + \varepsilon S_{n+1} : S, S_{n+1}) \leq (\alpha - n - 1)|\log \varepsilon| + \text{const.}$
- Standard properties: suffices to bound $\chi(Y_1 - \mathbb{E}_\varepsilon(Y_1) + \varepsilon S_{n+1})$.
- Change of variable formula: $\log \varepsilon + \chi(\varepsilon^{-1}(Y_1 - \mathbb{E}_\varepsilon(Y_1)) + S_{n+1})$.
- Latter term is controlled by $\varepsilon^{-1} d_2(Y_1; X)(\varepsilon)$.

The Determinant Condition

- To complete the proof that $\mathcal{L}\mathbb{F}O_M$ and $\mathcal{L}\mathbb{F}O_{2N}^J$ are strongly 1-bounded, it now suffices to show that $\partial F(u)^*\partial F(u)$ is a determinant class operator.
- First we have to face something that has been swept under the rug until now.
- **Obstacle:** the canonical generators u_{ij} are self-adjoint when $Q = I_M$, but *not* when $Q = J_{2N}$.
- **Solution:** The relation $u = -J\overline{u}J$ gives a decomposition

$$u = \tau_a \otimes A^u + \tau_b \otimes B^u + \tau_c \otimes C^u + \tau_d \otimes D^u.$$

- Here, $\overline{A^u} = A^u \in M_N(\mathbb{C}) \otimes C^*\mathbb{F}O_{2N}^J$ *et cetera*, and $\tau_a = I_2$, $\tau_b = i\sigma_y$, $\tau_c = i\sigma_z$, $\tau_d = i\sigma_x$ (Pauli matrices).
- This defines self-adjoint generators $a_{ij}^u, \ldots, d_{ij}^u$ and the decomposition has good algebraic properties.

The Determinant Condition II

- Computing $\partial F(u)$ for $Q = I_M$ is straightforward, the case $Q = J_{2N}$ is a bit more involved.
- The results can be rewritten entirely in terms of quantum group theoretic data coming from $\mathbb{F}O(Q)$.
- $\partial F(u)^* \partial F(u)$ turns out to be the same for $Q = I_M, J_{2N}$.
- It is unitarily conjugate to $4(1 + \mathrm{Re}[\vartheta \otimes 1])$, with ϑ the *edge reversing operator* on the *quantum Cayley graph* of $\mathbb{F}O(Q)$.
- Quantum Cayley graphs were introduced and studied by Vergnioux.
- It so happens that $1 + \mathrm{Re}[\vartheta \otimes 1]$ is a determinant class operator for essentially all Q.

Quantum Cayley Graphs

- Recall the operator $U(x.\xi_0) = S(x).\xi_0$ where S is the antipode, and write Σ for the tensor flip map.
- There is a dense subspace $\oplus_{v \in \mathrm{Irr}(Q)} B(\mathcal{H}_v) \subset \ell^2 \mathbb{F}O(Q)$ and two orthogonal projections $p_{0,1}$ such that $p_0 p_1 = 0$, $Up_1 = p_1 U$, $p_0 \ell^2 \mathbb{F}O(Q) = \mathbb{C}\xi_0$, and $p_1 \ell^2 \mathbb{F}O(Q) = B(\mathcal{H}_u) \cong M_N(\mathbb{C})$.
- Now define the vertex Hilbert space to be $\mathcal{H}_Q = \ell^2 \mathbb{F}O(Q)$ and the edge Hilbert space to be $\mathcal{K}_Q = \ell^2 \mathbb{F}O(Q) \otimes p_1 \ell^2 \mathbb{F}O(Q)$.
- Introduce two operators:
  - the boundary operator $E \in B(\mathcal{K}_Q, \mathcal{H}_Q \otimes \mathcal{H}_Q)$, given by restricting the multiplicative unitary V to $\mathcal{K}_Q$;
  - the edge-reversing operator $\vartheta \in B(\mathcal{K}_Q)$, given by $\vartheta = \Sigma(1 \otimes U) V (U \otimes U) \Sigma$.
- The latter is unitary, but not necessarily involutive (in spite of what the name may suggest).
- The quadruple $(\mathcal{H}_Q, \mathcal{K}_Q, E, \vartheta)$ is the quantum Cayley graph of $\mathbb{F}O(Q)$.

Quantum Cayley Graphs II

- Let us briefly discuss the relation to the classical Cayley graph of a group.
- Let G be a group and S a finite subset such that $S = S^{-1}$ and $e \notin S$. Set $p_0 = 1_{\{e\}}$ and $p_1 = 1_S$.
- Then the Hilbert space of vertices is $\ell^2 G = \overline{\text{span}}\{\delta_g\}$, and the Hilbert space of edges is $\ell^2 G \otimes \ell^2 S$.
- One can compute that for $g \in G$ and $s \in S$: $E(\delta_g \otimes \delta_s) = \delta_g \otimes \delta_{gs}$ and $\vartheta(\delta_g \otimes \delta_s) = \delta_{gs} \otimes \delta_{s^{-1}}$.
- So a pair $\delta_g \otimes \delta_s$ should be thought of as an edge from g to gs, i.e., it represents the origin and direction of an edge.

Quantum Cayley Graphs III

Theorem (Brannan-Vergnioux)

The operator $1+\mathrm{Re}[\vartheta]$, *viewed as an element of* $U\mathcal{L}\mathbb{F}O(Q)U\otimes B(p_1\mathcal{H}_Q)$, *is of determinant class with respect to* $h\otimes\mathrm{Tr}$.

- Write $K_g^{\pm}=\mathrm{Ker}(\vartheta\pm 1)$. There is an involutive unitary W such that $W\vartheta W=\vartheta^*$, and set $K_s=\mathrm{Ker}(W-1)$.
- Have the orthogonal decomposition $\mathcal{K}_Q=L\oplus L^{\perp}$, with $L=K_s\cap(K_g^+)^{\perp}\cap(K_g^-)^{\perp}$ the relevant subspace.
- On L, ϑ can be brought into the form $-r$, with r a weighted right shift (positive weights, at most 1).
- A $h\otimes\mathrm{Tr}$ is transformed too, what needs to be shown is that for a weighted right shift R on $\ell^2(\mathbb{N})$ such that δ_0 is in the range of $\sqrt{1-(\mathrm{Re}[R])^2}$, $\langle\delta_0,(1-(\mathrm{Re}[R])^2)^{-1}\log_+(1-\mathrm{Re}[R])\delta_0\rangle$ is finite.
- This follows by comparison with the standard right shift.

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