Interpolation between random matrices and their free limit with the help of free stochastic calculus

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Based on the following works,

On the operator norm of non-commutative polynomials in deterministic matrices and iid GUE matrices (joint work with B. Collins and A. Guionnet).

Asymptotic expansion of smooth functions in polynomials in deterministic matrices and iid GUE matrices.

Original motivation

Problem

Given a family $X^N = (X_1^N, \dots, X_d^N)$ *of self-adjoint random matrices,* P *a noncommutative polynomial, how does the operator norm of* $P(X^N)$ *behaves asymptotically? I.e. can we compute* $\lim_{N\to\infty} \left\| P(X^N) \right\|$*?*

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A necessary assumption

There exists a family $x = (x_1, \dots, x_d)$ *of self-adjoint elements of a* $\mathcal{C}^*$*-algebra* $\mathcal{A}$ *endowed with a faithful trace* τ *such that almost surely, the* X^N *converges in distribution towards* x*. That for any noncommutative polynomial* Q*,*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}\left(Q(X^N) \right) = \tau(Q(x)).$$

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Problem

Given a family $X^N = (X_1^N, \dots, X_d^N)$ *of random matrices,* P *a noncommutative polynomial, can we prove that almost surely:*

$$\lim_{N\to\infty} \left\|P(X^N)\right\| = \|P(x)\|?$$

The easy part of the solution

- Thanks to measure concentration estimates, we get that almost surely:

$$\lim_{N\to\infty} \left(\left\| P(X^N) \right\| - \mathbb{E}\left[\left\| P(X^N) \right\| \right] \right) = 0.$$

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- Since $\left\| P(X^N) \right\| = \left\| P(X^N)^* P(X^N) \right\|^{1/2}$, we can replace P by P^*P and thus assume that P is self-adjoint and non-negative. If f is a non-negative continuous function such that

$$\forall x \in (\|P(x)\| - \varepsilon, \|P(x)\| + \varepsilon),\ \ f(x) > 0,$$
$$\forall x \notin (\|P(x)\| - \varepsilon, \|P(x)\| + \varepsilon),\ \ f(x) = 0.$$

Then almost surely $\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}\left(f(P(X^N)) \right) = \tau\Big(f(P(x)) \Big) > 0$ since τ is faithful.

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Then almost surely $\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}\left(f(P(X^N)) \right) = \tau\left(f(P(x)) \right) > 0$ since τ is faithful. Thus almost surely, for any $\varepsilon > 0$, there is an eigenvalue of $P(X^N)$ in $(\|P(x)\| - \varepsilon, \|P(x)\| + \varepsilon)$. Hence

$$\liminf_{N\to\infty} \left\| P(X^N) \right\| \geq \|P(x)\| .$$

And by Fatou's lemma,

$$\liminf_{N\to\infty} \mathbb{E}\left[\left\| P(X^N) \right\| \right] \geq \mathbb{E}\left[\liminf_{N\to\infty} \left\| P(X^N) \right\| \right] \geq \|P(x)\| .$$

The hard part of the solution

let f_ε be a non-negative function equal to 1 on the interval $[\|P(x)\| + \varepsilon, \infty)$, 0 on $(-\infty, \|P(x)\|]$, then for any $\alpha > 0$,

$$\begin{aligned}
\mathbb{E}\left[\left\|P(X^N)\right\|\right] - \|P(x)\| &\leq \alpha + \int_\alpha^\infty \mathbb{P}\left(\left\|P(X^N)\right\| - \|P(x)\| \geq \varepsilon\right)\, d\varepsilon \\
&\leq \alpha + \int_\alpha^\infty \mathbb{P}\left(\mathrm{Tr}\left(f_\varepsilon(P(X^N))\right) \geq 1\right)\, d\varepsilon \\
&\leq \alpha + \int_\alpha^\infty \mathbb{E}\left[\mathrm{Tr}\left(f_\varepsilon(P(X^N))\right)\right]\, d\varepsilon.
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So to prove that

$$\limsup_{N\to\infty} \mathbb{E}\left[\left\|P(X^N)\right\|\right] \leq \|P(x)\|,$$

we need to prove that if the support of a smooth function g is disjoint from the spectrum of $P(x)$, then

$$\lim_{N\to\infty} \mathbb{E}\left[\mathrm{Tr}\left(g(P(X^N))\right)\right] = 0.$$

Definition

We say that X^N is a GUE random matrix of size N if.

- *It is a self-adjoint random matrix.*
- *Diagonal coefficents are independent centered gaussian random variable of variance $1/N$.*
- *Upper diagonal coefficents are independent complex centered gaussian random variable of variance $1/N$.*

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Theorem (D. Voiculescu, 1991)

Let $X^N = (X_1^N, \ldots, X_d^N)$ be independent GUE matrices, $x = (x_1, \ldots, x_d)$ be a system of free semicircular variables. Then almost surely X^N converges in distribution towards x. That is almost surely for any noncommutative polynomial P,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}\Big(P(X^N)\Big) = \tau\Big(P(x)\Big) .$$

The convergence of the operator norm

- It is easy to show that

$$\lim_{N\to\infty} \left(\left\| P(X^N) \right\| - \mathbb{E}\left[\left\| P(X^N) \right\| \right] \right) = 0.$$

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- And finally, in 2005, Haagerup and Thorbjørnsen proved that for g smooth enough.

$$\mathbb{E}\left[\frac{1}{N} \mathrm{Tr}\left(g\left(P(X^N) \right) \right) \right] = \tau(g(P(x))) + \mathcal{O}(N^{-2}),$$

which proves the convergence of the operator norm.

Previous works

By strong convergence we mean the convergence of the operator norm of any polynomials.

- Haagerup, Thorbjørnsen (2005): strong convergence of a familly of independent GUE matrices.
- Schultz (2005): strong convergence of a familly of independent GOE or GSE matrices.
- Capitaine, Donati-Martin (2006): strong convergence of a familly of independent Wigner or Wishart matrices under some assumptions.
- Anderson (2013): strong convergence of a familly of independent Wigner matrices under very weak hypothesis.
- Male (2012): strong convergence of a familly of deterministic and independent GUE matrices.
- Belinschi, Capitaine (2016): strong convergence of a family of independent deterministic and Wigner matrices
- Collins, Male (2014): strong convergence of a familly of deterministic and independent unitary Haar matrices.
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Except for the unitary case, those proofs rely heavily on the so-called linearization trick, which allows to relate the spectrum of a polynomial of degree d with coefficients in $\mathbb{C}$ by a polynomial of degree 1 with coefficients in $\mathbb{M}_{k(d)}(\mathbb{C})$. Unfortunately it makes retrieving good quantitative estimates tricky, hence we focused on finding a different proof which does not use the linearization trick.

Stein's method in classical probability

If Z is a centered Gaussian variable of variance 1 if and only if for any smooth function f,

$$\mathbb{E}\left[Zf(Z)\right] = \mathbb{E}\left[f'(Z)\right]. \tag{1}$$

We set $\mathcal{A} : f \in \mathcal{C}^\infty(\mathbb{R}) \mapsto (x \to f'(x) - xf(x))$, then intuitively the distribution of a random variable Y should be close from a Gaussian one if for any function f, $\mathbb{E}[\mathcal{A}f(Y)]$ is small.

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Let $(X_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process started in Y, that is

$$\forall t \geq 0, \quad X_t = Y - \frac{1}{2}\int_0^t X_s ds + B_t.$$

Then,

$$\mathbb{E}\left[f(Z)\right] - \mathbb{E}\left[f(Y)\right] = \frac{1}{2}\int_0^{\infty} e^{-t/2}\mathbb{E}\left[\mathcal{A}g_t(Y)\right] dt,$$

Hence if $\mathbb{E}[\mathcal{A}f(Y)]$ is small for any smooth f, we do get that the distribution of Y is close from a Gaussian one.

Stein's method in Random Matrix Theory

If X is a GUE matrix of size N, then with $\tau_N = N^{-1}\,\mathrm{Tr}$, for any $p \geq 0$,

$$\mathbb{E}\left[\tau_N\left(X^{p+1}\right) - \sum_{i=1}^{p-1}\tau_N\left(X^i\right)\tau_N\left(X^{p-1-i}\right)\right] = 0. \tag{2}$$

If Y is a Wigner random matrix of size N, then

$$\mathbb{E}\left[\tau_N\left(Y^{p+1}\right) - \sum_{i=1}^{p-1}\tau_N\left(Y^i\right)\tau_N\left(Y^{p-1-i}\right)\right] = \mathcal{O}(N^{-1/2}). \tag{3}$$

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Let $(H_t)_{t\geq 0}$ be a Hermitian Brownian motion. Then we can define a Hermitian Ornstein-Uhlenbeck process started in Y by

$$X_t = Y - \frac{1}{2}\int_0^t X_s ds + H_t.$$

Then we have

$$\mathbb{E}\left[\tau_N\left(X^p\right)\right] - \mathbb{E}\left[\tau_N\left(Y^p\right)\right] = \frac{l}{2}\int_0^\infty \mathbb{E}\left[\tau_N\left(X_t^p\right) - \sum_{i=1}^{p-2} \tau_N\left(X_t^i\right)\tau_N\left(X_t^{p-2-i}\right)\right] dt.$$

The limit of the moments of GUE random matrices converges towards the moments of the semicircle distribution. Hence this generalizes to Wigner matrices too.

A free probability version of Stein's Method

If X is a GUE matrix of size N, then thanks to equation (2) as well as usual measure concentration estimates, for any $p \geq 0$,

$$\mathbb{E}\left[\tau_N\left(X^{p+1}\right)\right] - \sum_{i=1}^{p-1} \mathbb{E}\left[\tau_N\left(X^i\right)\right] \mathbb{E}\left[\tau_N\left(X^{p-1-i}\right)\right] = \mathcal{O}(N^-2). \tag{4}$$

If x is a semicircular variable, then

$$\tau\left(x^{p+1}\right) - \sum_{i=1}^{p-1} \tau\left(x^i\right) \tau\left(x^{p-1-i}\right) = 0. \tag{5}$$

It is known that if P is a polynomial, $\mathbb{E}\left[\tau_N\left(P(X)\right)\right]$ converges towards $\tau(P(x))$. But how can we estimate the difference between those two quantities?

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It is known that if P is a polynomial, $\mathbb{E}\left[\tau_N\left(P(X)\right)\right]$ converges towards $\tau(P(x))$. But how can we estimate the difference between those two quantities?

Once again there exist a free equivalent to the usual Brownian motion, the free Brownian motion, with whom we can build a free Ornstein-Uhlenbeck process started in X. There also exist a theory of free stochastic calculus and free Markov process, and the generator of the free Ornstein-Uhlenbeck process matches with equation (4) and (5).

In all generality though, we need to work with several GUE matrix, and smooth functions instead of moments.

A strong convergence theorem

By using this method we end up proving again that

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(g\left(P(X^N)\right)\right)\right] = \tau(g(P(x))) + \mathcal{O}(N^{-2}),$$

with an explicit estimate of the last term. It has several corollaries, but the most important one is the following:

Theorem (Collins, Guionnet, P., 2019)

Let the following objects be given:

- X^N *independent GUE matrices of size* N,
- Z^N *deterministic matrices which converge strongly in distribution towards a family* z,
- x *a system of free semicircular variables,*
- Y^M *deterministic matrices of size* M, *which converge strongly in distribution towards a family* y.

Then, the following holds true:

- *The family* (X^N, Z^N) *converges strongly towards* (x, z).
- *If* $M = o(N^{1/3})$, *almost surely,* $(X^N \otimes I_M, I_N \otimes Y^M)$ *converges strongly in distribution towards* $(x \otimes 1, 1 \otimes y)$.

In particular, the strong convergence of the family (X^N, Z^N) was already proved by Camille Male in 2012.

A finite order Taylor expansion

- Naturally one can wonder what happens at the next order. More precisely, could we write this expectation as a finite order Taylor expansion. That is, can we prove that for any k, if f is smooth enough, there exist deterministic constants $\alpha_i^P(f)$ such that

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(f(P(X_1^N,\dots,X_d^N))\right)\right] = \sum_{i=0}^{k}\frac{\alpha_i^P(f)}{N^{2i}} + \mathcal{O}(N^{-2k-2}).$$

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- In the case where f is a polynomial and $d=1$, Harer and Zagier gave a positive answer in 1986. They proved that given $\mathcal{M}_g^k$ the number of maps of genus g, one vertex and $k/2$ edges.

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left((X_1^N)^k\right)\right] = \sum_{g\in\mathbb{N}}\frac{1}{N^{2g}}\mathcal{M}_g^k.$$

- Haagerup and Thorbjørnsen gave a positive answer in 2010 for the specific case of a single GUE matrix.

The main result: the case of independent GUE matrices

Theorem (P., 2020)

Let the following objects be given,

- $X^N = (X_1^N, \dots, X_d^N)$ *independent GUE matrices in* $\mathbb{M}_N(\mathbb{C})$,
- P *a self-adjoint polynomial,*
- $f \in \mathcal{C}^{4k+6}(\mathbb{R})$.

Then there exist determinitic constants $(\alpha_i^P(f))_{i\in\mathbb{N}}$ *such that,*

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}_N\left(f(P(X^N))\right)\right] = \sum_{0\leq i\leq k} \frac{1}{N^{2i}}\alpha_i^P(f) + \mathcal{O}(N^{-2(k+1)}).$$

Besides, if the support of f *and the spectrum of* $P(x)$ *are disjoint, then for any* i, $\alpha_i^P(f) = 0$.

Consequences

Corollary (P., 2020)

Let X^N be independent GUE matrices of size N, x be a free semicircular system and P a self-adjoint polynomial. Given $\alpha < 1/2$, almost surely for N large enough,

$$\sigma\left(P(X^N)\right) \subset \sigma\left(P(x)\right) + N^{-\alpha},$$

where $\sigma(X)$ is the spectrum of X.

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Corollary (P., 2020)

Let X^N be a vector of independent GUE matrices of size N, x be a free semicircular system and P a polynomial. Then there exist a constant C such that for any $\alpha < 1/2$,

$$\mathbb{P}\left(N^{-\alpha}\left(\left\|P(X^N)\right\| - \|P(x)\|\right) \geq \delta\right) \leq C \times N \times e^{-\delta \wedge 1 \times N^{1/2-\alpha}}.$$

Idea of the proof

We want to show the following formula:

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}_N\left(f(P(X^N))\right)\right] = \sum_{0\le i\le k} \frac{1}{N^{2i}}\alpha_i^P(f) + \mathcal{O}(N^{-2(k+1)}).$$

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- Secondly we set Q a polynomial in $X_1, \dots, X_d, Y_1, \dots, Y_p$ and $X_t^N = e^{-t/2}X^N + (1-e^{-t})^{1/2}x$ (which is somehow our free Ornstein-Uhlenbeck process), then given a system of free semicircular y, for any polynomial Q,

$$\mathbb{E}\left[\tau_N\Big(Q(X^N, y)\Big)\right] = \tau\Big(Q(x,y)\Big) - \int_0^\infty \mathbb{E}\left[\frac{d}{dt}\tau_N\Big(Q(X_t^N, y)\Big)\right] dt.$$

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- Then we show that there is a deterministic operator $T_{t,y}$ on the space of polynomials such that
$$\frac{d}{dt}\tau_N\Big(Q(X_t^N,y)\Big) = \frac{1}{N^2}\tau_N\Big(T_{t,y}(Q)(X_t^N,y)\Big).$$
- We can view the polynomial $T_{t,y}(Q)$ as a polynomial in (X^N,x,y) and reiterate the process.