Commutator estimates in W^*-algebras and applications

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April 25, 2020

J. von Neumann, *Some matrix inequalities and metrization of matric-space*, Rev. Tomsk Univ. **1** (1937), 286–300.

In 1937, J. von Neumann showed that if $\|\cdot\|_E$ is a symmetric norm on $\mathbb{R}^n$ then one can define a norm on the space of $n \times n$ matrices by

$$\|A\|_E = \|(s_1(A), \ldots, s_n(A))\|_E ,$$

where $s_1(A), \ldots, s_n(A)$ are the singular values of A (i.e. the eigenvalues of $(A^*A)^{1/2}$) in decreasing order. Infinite-dimensional development of this pioneering result by von Neumann is in highly influential books by R. Schatten, Gohberg&Krein and B. Simon. All classical Banach space geometry (books by Lindenstrauss&Tzafriri) is strongly allied with this object.

N. Kalton and F.S.,*Symmetric norms and spaces of operators,* J. Reine Angew. Math. **621** (2008), 81–121.

Symmetric Banach sequence space E is a Banach ideal of the space $\ell_\infty = \ell_\infty(\mathbb{N})$ whose norm is invariant under permutations. Let H be a Hilbert space. We define an associated *Schatten ideal* $\mathcal{S}_E \subset B(H)$ by $T \in \mathcal{S}_E \iff (s_n(T))_{n=1}^\infty \in E$ with a (quasi-)norm $\|T\|_{\mathcal{S}_E} = \|(s_n(T))_{n=1}^\infty\|_E$. Recall, that a Banach space $(\mathcal{E}, \|\cdot\|_\mathcal{E})$ which a linear subspace in $B(H)$ is said to be a Banach ideal in $B(H)$ if its norm $\|\cdot\|_\mathcal{E}$ satisfies the following estimates

$$\|XY\|_\mathcal{E} \le \|X\|_\mathcal{E}\|Y\|_\infty,\ \|YX\|_\mathcal{E} \le \|X\|_\mathcal{E}\|Y\|_\infty,\ \forall X \in \mathcal{E},\ Y \in B(H).$$

MAIN RESULT of [KS]: $(\mathcal{S}_E, \|\cdot\|_{\mathcal{S}_E})$ is a Banach ideal in $B(H)$.

J.W. Calkin, *Two-sided ideals and congruences in the ring of bounded operators in Hilbert space*, Ann Math. **42** (1941), 839-873.

From Introduction The developments of the present paper center around the observation that the ring $B(H)$ of bounded everywhere defined operators in Hilbert space contains **non-trivial two-sided ideals**. This fact, which has escaped all but oblique notice in the development of the theory of operators, is of course **fundamental** from the point of view of algebra and at the same time differentiates $B(H)$ sharply from the ring of all linear operators over a unitary space with finite dimension number.

Calkin algebra

Let H be a separable Hilbert space and let $\mathcal{K}$ be the C^*-algebra of all compact operators on H. $\mathcal{K}$ is a noncommutative analogue of the space c_0 of all vanishing sequences. The ideal $\mathcal{K}$ is closed in $B(H)$, which is a noncommutative analogue of the algebra ℓ_∞ of all bounded sequences. The quotient C^*-algebra $\mathcal{C} = B(H)/\mathcal{K}$ is called the Calkin algebra. It is a noncommutative analogue of the quotient algebra ℓ_∞/c_0. "*The Calkin algebra is important because it is the repository of all asymptotic information about operators on H*". (Arveson).

Theorem 2.9 from J.W. Calkin's paper

THEOREM 2.9. *Let $\mathcal{J}$ be an ideal in $B(H)$, $\mathcal{J} \neq B(H)$. Then the center $B(H)/\mathcal{J}$, that is the set of all elements from $B(H)/\mathcal{J}$ which commute with every element of $B(H)/\mathcal{J}$, is the set of all elements $\lambda \mathbf{1}$, where λ is a complex number.*

THE GIST: If $A \in B(H)$ is such that $AB - BA \in \mathcal{J}$ for all $B \in B(H)$, then $A = T + \lambda \mathbf{1}$ for some $T \in \mathcal{J}$ and $\lambda \in \mathbb{C}$.

RESTATEMENT: $D(B(H), \mathcal{J}) := \{T \in B(H) : [T, S] := TS - ST \in \mathcal{J} \ \forall S \in B(H)\} = \mathcal{J} + \mathbb{C}\mathbf{1}$

Some notations

Let $\mathcal{A}$ be a complex algebra. For $x, y \in \mathcal{A}$, the *commutator* $[x, y]$ is defined by setting

$$[x, y] := xy - yx.$$

A linear map $\delta : \mathcal{A} \to \mathcal{A}$ is called a derivation if

$$\delta(xy) = \delta(x)y + x\delta(y),\ \forall x, y \in \mathcal{A}.$$

If $w \in \mathcal{A}$, then the map $\delta_w : \mathcal{A} \to \mathcal{A}$ given by $\delta_w(x) = [w, x]$, $x \in \mathcal{A}$ is a derivation. A derivation of this form is called inner.

Not every derivation on a C^*-algebra is inner. Let $\mathcal{K}$ be the algebra of compact operators on H.

Example (Sakai, C^*-algebras and W^*-algebras, Springer, 1971, see also Ber-Huang-Levitina-S.-JFA-2017 fore more examples)

Take an arbitrary element $w \in B(H)$ which is not in $\mathcal{K} + \mathbb{C}\mathbf{1}$. Consider $\delta_w : B(H) \to B(H)$. It is trivial that $\delta_w(x) \in \mathcal{K}$ for any $x \in \mathcal{K}$. That is $\delta := \delta_w|_{\mathcal{K}}$ is a derivation on $\mathcal{K}$. However, δ is not inner on $\mathcal{K}$! Indeed, suppose there exists an element $v \in \mathcal{K}$ such that $\delta = \delta_v$. Then since $\delta_v = \delta_w$ on $\mathcal{K}$, we immediately conclude that $v - w$ belongs to the center $Z(\mathcal{K})$. Hence, $w = v + \lambda\mathbf{1}$, which is a contradictions, since $w \notin \mathcal{K} + \mathbb{C}\mathbf{1}$ and $v \in \mathcal{K}$.

However, we claim that every derivation $\delta_w : B(H) \to \mathcal{K}$ is inner (see Johnson–Parrott–Popa Theorem (Popa, JFA, 1985) and Ber–Huang–Levitina–S. for more general results).
We provide here a very short proof of a slightly more general result. Let M be a von Neumann algebra, $d \in M$ and $[d, M] \subset J$, where J is **closed in the uniform norm**. Obviously, $du - ud \in J$ for any unitary $u \in M$. Equivalently $u^* du - d \in J$ for any unitary $u \in M$. By the DAT, there exists a convex combination of $u_i^* du_i$ converging to $z \in Z(M)$ in norm. Obviously, the same convex combination of $u_i^* du_i - d$ converging to $z - d$ in norm. Since J is norm closed and since $u_i^* xu_i - d \in J$, we obtain that the uniform norm limit $z - d \in J$ ($\approx$ Theorem 2.9). Finally, $[d, \cdot] = [d - z, \cdot]$. That is $\delta_d = \delta_{d-z}$.

M.J. Hoffman, *Essential commutants and multiplier ideals*, Indiana Univ. Math. J. **30** (1981), no. 6, 859–869.

Replace $B(H)$ with an arbitrary ideal $\mathcal{I}$. That is, fix two self-adjoint ideals $\mathcal{J}, \mathcal{I}$ in $B(H)$. We set

$$\mathcal{J} : \mathcal{I} = \{x \in B(H) : \; x\mathcal{I} \subset \mathcal{J}\}$$

and

$$D(\mathcal{I}, \mathcal{J}) = \{T \in B(H) : \; [T, S] \in \mathcal{J}, \; \forall S \in \mathcal{I}\}.$$

MAIN RESULT: $D(\mathcal{I}, \mathcal{J}) = \mathcal{J} : \mathcal{I} + \mathbb{C}\mathbf{1}$. If $\mathcal{I} = B(H)$, then $\mathcal{J} : \mathcal{I} = \mathcal{J} : B(H) = \mathcal{J}$ and the assertion above yields Calkin's Th.2.9.

Theorem 4.1.6. from Sakai's book "C^* and W^*-algebras"

Theorem

Let δ be a derivation on a W^*-algebra $\mathcal{M}$. Then δ is inner, namely there exists an element $a \in \mathcal{M}$ such that $\delta(x) = [a, x]$, $x \in \mathcal{M}$. Moreover, we can choose such an element a as follows: $\|a\| \leq \|\delta\|$.

Remark

Observe that what this theorem actually says is the following: given $a \in \mathcal{M}$, there exists an element $c \in Z(\mathcal{M})$ such that $\|a - c\| \leq \|\delta_a\|$. Indeed, if for $a, b \in \mathcal{M}$, the inner derivations δ_a and δ_b coincide, then we necessarily have $\delta_{a-b} = 0$ on $\mathcal{M}$, and therefore $a - b \in \mathcal{M}'$. The latter implies immediately that $a - b \in Z(\mathcal{M})$.

Questions motivated by Calkin and Hoffman's work

Let $\mathcal{M}$ be a W^*-algebra and let $\mathcal{I}$ be an ideal in $\mathcal{M}$. We fix this notation.

Motivated by Calkin's Th.2.9, we ask: *"Let $\pi : \mathcal{M} \to \mathcal{M}/\mathcal{I}$ be a canonical epimorphism. Do we have*

$$\pi^{-1}(center(\mathcal{M}/\mathcal{I})) = Z(\mathcal{M}) + \mathcal{I}?"$$

Motivated by Hoffman's "derivation viewpoint" on Calkin's Th.2.9, we ask: *"Let $\delta : \mathcal{M} \to \mathcal{I}$ be a derivation. Does there exist an element $a \in \mathcal{I}$, such that $\delta(\cdot) = \delta_a(\cdot) = [a, \cdot]$?"*

Remember, our ideal $\mathcal{I}$ is not necessarily uniform norm closed. The DAT is not applicable! Calkin's and Hoffman's techniques (which are heavily $B(H)$-type) are not applicable!

Question motivated by Sakai's Theorem 4.1.6

Let $\mathcal{M}$ be a W^*-algebra and let $Z(\mathcal{M})$ be the center of $\mathcal{M}$. Fix $a \in \mathcal{M}$ and consider the inner derivation δ_a on $\mathcal{M}$. It follows from Sakai's Theorem 4.1.6, that there exists $c \in Z(\mathcal{M})$ such that $\|\delta_a\| \geq \|a - c\|_{\mathcal{M}}$. In view of this result, it is natural to ask whether there exists an element $y \in \mathcal{M}$ with $\|y\| \leq 1$ and $c \in Z(\mathcal{M})$ such that $|[a, y]| \geq |a - c|$?
Our main result (next slide) answers this as well as preceding questions.

Theorem (Ber-S)

Let $\mathcal{M}$ be a W^*-algebra and let $a = a^* \in \mathcal{M}$.

1. *There exists $c_0 = c_0^* \in Z(\mathcal{M})$, so that for any $\varepsilon > 0$ there exists $u_\varepsilon = u_\varepsilon^* \in U(\mathcal{M})$ such that*

$$|[a, u_\varepsilon]| \geq (1 - \varepsilon)|a - c_0|. \tag{1}$$

2. *If $\mathcal{M}$ is a finite W^*-algebra or else a purely infinite σ-finite W^*-algebra, then there exists $c_0 = c_0^* \in Z(\mathcal{M})$ and $u_0 = u_0^* \in U(\mathcal{M})$, such that*

$$|[a, u_0]| = u_0^* |a - c_0| u_0 + |a - c_0|, \tag{2}$$

where $U(\mathcal{M})$ is the group of all unitary elements in $\mathcal{M}$;

Sharpness

Observe that the equality (2) trivially yields the estimate (1) even for the case $\varepsilon = 0$. Nevertheless, the result of (i) is still sharp. Indeed, if $\mathcal{M}$ is an infinite semifinite σ-finite factor, then there exists a self-adjoint element $a \in \mathcal{M}$ such that for every $\lambda \in \mathbb{C}$ and $u \in U(\mathcal{M})$ the inequality $|[a, u]| \geq |a - \lambda \mathbf{1}|$ fails. Hence, the multiplier $(1 - \varepsilon)$ in the part *(i)* of Theorem **??** can not be omitted.

First version of extension of Calkin's Th.2.9

Our Main Theorem yields a completely different proof of Calkin's Th.2.9. It also answers the questions stated above.

Corollary (Ber-S)

Let $\mathcal{M}$ be a W^*-algebra and let $\mathcal{I}$ be an ideal in $\mathcal{M}$. Let $\delta : \mathcal{M} \to \mathcal{I}$ be a derivation. Then there exists an element $a \in \mathcal{I}$, such that $\delta = \delta_a = [a, \cdot]$.

Proof.

Since δ is a derivation on a W^*-algebra, it is necessarily inner (Sakai's [Theorem 4.1.6]). Thus, there exists an element $d \in \mathcal{M}$, such that $\delta(\cdot) = \delta_d(\cdot) = [d, \cdot]$. It follows from our hypothesis that $[d, \mathcal{M}] \subseteq \mathcal{I}$.

Without loss of generality, let d be self-adjoint. It follows now from Ber-S Theorem, that there exist $c \in Z(\mathcal{M})$ and $u \in U(\mathcal{M})$, such that $|[d, u]| \geq 1/2|d - c|$. This implies obtain $d - c \in \mathcal{I}$. Setting $a := d - c$, we deduce that $a \in \mathcal{I}$ and $\delta = [a, \cdot]$. □

Second version of extension of Calkin's Th.2.9

Corollary

Let $\mathcal{M}$ be a W^*-algebra, let $\mathcal{I}$ be an ideal in $\mathcal{M}$ and let $\pi : \mathcal{M} \to \mathcal{M}/\mathcal{I}$ be a canonical epimorphism. Then, $\pi^{-1}(center(\mathcal{M}/\mathcal{I})) = Z(\mathcal{M}) + \mathcal{I}$.

Proof.

Let $a \in \pi^{-1}(center(\mathcal{M}/\mathcal{I}))$. Then $[a, x] = ax - xa \in \mathcal{I}$ for any $x \in \mathcal{M}$. By the preceding Corollary applied to δ_a, we obtain $a + c \in \mathcal{I}$ for some $c \in Z(\mathcal{M})$. Therefore $a \in Z(\mathcal{M}) + \mathcal{I}$, that is $\pi^{-1}(center(\mathcal{M}/\mathcal{I})) \subset Z(\mathcal{M}) + \mathcal{I}$. The converse inclusion is trivial. □

Setting up the scene for Hoffman-type results

Let us fix a W^*-algebra $\mathcal{M}$ and two self-adjoint ideals $\mathcal{I}, \mathcal{J}$ in $\mathcal{M}$. We set

$$\mathcal{J} : \mathcal{I} = \{x \in \mathcal{M} : \ x\mathcal{I} \subset \mathcal{J}\}$$

and

$$D(\mathcal{I}, \mathcal{J}) = \{x \in \mathcal{M} : \ [x, y] \in \mathcal{J}, \ \forall y \in \mathcal{I}\}.$$

Observe that $\mathcal{J} : \mathcal{I}$ is an ideal in $\mathcal{M}$. In particular, $(\mathcal{J} : \mathcal{I})^* = \mathcal{J} : \mathcal{I} = \{x \in \mathcal{M} : \ \mathcal{I}x \subset \mathcal{J}\}$.

Extending Hoffman's results

Corollary

For any W^*-algebra $\mathcal{M}$ and any ideals $\mathcal{I}, \mathcal{J}$ in $\mathcal{M}$ we have $D(\mathcal{I}, \mathcal{J}) = \mathcal{J} : \mathcal{I} + Z(\mathcal{M})$.

Applications to normed ideals.

Definition

A linear subspace $\mathcal{I}$ in the von Neumann algebra $\mathcal{M}$ equipped with a norm $\|\cdot\|_{\mathcal{I}}$ is said to be *a symmetric operator ideal* if
(1) $\|S\|_{\mathcal{I}} \geq \|S\|_{\infty}$ for all $S \in \mathcal{I}$,
(2) $\|S^*\|_{\mathcal{I}} = \|S\|_{\mathcal{I}}$ for all $S \in \mathcal{I}$,
(3) $\|ASB\|_{\mathcal{I}} \leq \|A\|_{\infty} \|S\|_{\mathcal{I}} \|B\|_{\infty}$ for all $S \in \mathcal{I}$, $A, B \in \mathcal{M}$.

Observe, that every symmetric operator ideal $\mathcal{I}$ is a two-sided ideal in $\mathcal{M}$, and therefore, it follows from $0 \leq S \leq T$ and $T \in \mathcal{I}$ that $S \in \mathcal{I}$ and $\|S\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}$.

Applications to normed ideals.

Corollary

Let $\mathcal{M}$ be a W^*-algebra, let $\mathcal{I}$ be a symmetric operator ideal in $\mathcal{M}$ and let $\delta : \mathcal{M} \to \mathcal{I}$ be a self-adjoint derivation. Then there exists an element $a \in \mathcal{I}$, satisfying the inequality $\|a\|_{\mathcal{I}} \leq \|\delta\|_{\mathcal{M}\to\mathcal{I}}$ and such that $\delta = \delta_a = [a, \cdot]$.

The main idea in finite-dimensional setting-1

Let $\mathcal{M}$ coincide with algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices. Fix $a = a^* \in M_n(\mathbb{C})$. The claim of Theorem (Ber-S)

$$|[a, u_0]| = u_0^*|a - c_0|u_0 + |a - c_0|,$$

is invariant under the action of inner $*$-automorphisms and so, we can assume that

$$a = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ 0 & 0 & \lambda_n \end{vmatrix} \in M_n(\mathbb{C}),$$

where $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$.

The main idea in finite-dimensional setting-2

Let the unitary matrix $u \in M_n(\mathbb{C})$ be counter-diagonal, that is

$$u = \begin{vmatrix} 0 & & 0 & 1 \\ 0 & & 1 & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 1 & 0 & & 0 \end{vmatrix},$$

and observe that

$$u^* a u = \begin{vmatrix} \lambda_n & 0 & 0 \\ 0 & \lambda_{n-1} & 0 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ 0 & 0 & \lambda_1 \end{vmatrix}.$$

The main idea in finite-dimensional setting-3

Therefore,

$$|[a,u]| = |u^*au - a| = \begin{vmatrix} |\lambda_n - \lambda_1| & 0 & 0 \\ 0 & |\lambda_{n-1} - \lambda_2| & 0 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ 0 & 0 & |\lambda_1 - \lambda_n| \end{vmatrix}.$$

If n is odd, then for all $1 \leq k \leq n$ we have

$$|\lambda_k - \lambda_{n+1-k}| = |\lambda_k - \lambda_0| + |\lambda_{n+1-k} - \lambda_0|$$

for $\lambda_0 = \lambda_{(n+1)/2}$.

The main idea in finite-dimensional setting-4

If n is even, then for all $1 \leq k \leq n$ we have

$$|\lambda_k - \lambda_{n+1-k}| = |\lambda_k - \lambda_0| + |\lambda_{n+1-k} - \lambda_0|$$

for every $\lambda_0 \in [\lambda_{n/2}, \lambda_{n/2+1}]$.
Therefore, for every $n \in \mathbb{N}$, we have

$$|[a, u]| = u^*|a - \lambda_0 \mathbf{1}|u + |a - \lambda_0 \mathbf{1}|.$$

This completes the proof.