Binary trees, operads and Dykema's T-transform

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Probabilistic Operator Algebra Seminar
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Aim

B-valued non-commutative probability space (A, E, B)

Formal multilinear function series Mul$[[B]]$

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots), \qquad \alpha_{k+1} : B^{\otimes k+1} \to B$$

Given $X \in A$ and $R_X \in$ Mul$[[B]]$, define $T_X \in$ Mul$[[B]]$

$$T_X \circ [I \cdot R_X] = R_X.$$

For $X, Y \in A$ free, Dykema[1] showed the twisted factorisation

$$T_{XY} \overset{(*)}{=} T_X \circ [T_Y \cdot I \cdot T_Y^{-1}] \cdot T_Y$$

We want to derive $(*)$ *from an operadic perspective...*

[1] K. Dykema, *Multilinear function series and transforms in free probability theory*, Adv Math 208 (2007) 351.

Moment-cumulant relations[2]

Non-commutative probability space (A, ϕ), $\phi : A \to \mathbb{C}$, $\phi(1_A) = 1$

$X \in A$

$$1 + M_X(z) = 1 + \sum_{n \geqslant 1} m_n(X) z^n \qquad R_X(z) = \sum_{n \geqslant 1} k_n(X) z^n$$

$$M_X(z) = R_X[z(1 + M_X(z))]$$

$$m_n(X) = \sum_{\pi \in NC_n} \prod_{V \in \pi} k_{|V|}(X).$$

$X, Y \in A$ free

$$R_{X+Y}(z) = R_X(z) + R_Y(z)$$

[2] A. Nica, R. Speicher, Lectures on the combinatorics of free probability, LMS, Lect. Note S. 335, CUP, 2006.

S- and T-transforms

Assume that $m_1(X) = k_1(X) = 1$.

$$S_X(z) = \frac{1}{z} R_X^{<-1>}(z) \qquad\qquad T_X(z) := \frac{1}{S_X(z)} = 1 + \sum_{n \geqslant 1} t_n(X) z^n$$

[Mastnak-Nica, 2010][3]:

$$\begin{aligned} z &= [zS_X(z)] \circ R_X(z) \\ &= R_X(z) S_X(R_X(z)) \;\Rightarrow\; \frac{R_X(z)}{z} = \Big[\frac{1}{S_X(z)} \circ R_X(z)\Big] = T[R_X(z)] \end{aligned}$$

$$\frac{R_X(z)}{z} = 1 + R'_X(z) = 1 + \sum_{n \geqslant 1} k_{n+1}(X) z^n$$

[3] M. Mastnak, A. Nica, *Hopf algebras and the logarithm of the S-transform in free probability*, TAMS 362 (2010), 3705.

S- and T-transforms

$$1 + R'_X(z) = T_X[z(1 + R'_X(z))]$$

$$k_{n+1}(X) = \sum_{\pi \in NC_n} \prod_{V \in \pi} t_{|V|}(X)$$

$X, Y \in A$ free

$$T_{XY}(z) = T_X(z) T_Y(z)$$

Remark: $X, Y \in A$ free

$$k_n(XY) = \sum_{\pi \in NC_n} k_\pi(X) k_{\mathsf{Kr}(\pi)}(Y), \qquad k_\pi(X) := \prod_{V \in \pi} k_{|V|}(X)$$

Kr(π) is the Kreweras complement of $\pi \in NC_n$.

Operator-valued setting[4]

Operator-valued non-commutative probability space (A, E, B):
A is an unital algebra with unital subalgebra $B \subset A$
conditional expectation $E : A \to B$

$$E(b) = b, \quad E(b_1 X b_2) = b_1 E(X) b_2, \qquad\qquad b, b_1, b_2 \in B,\ X \in A$$

$$E(X^n) = \sum_{\pi \in NC_n} k^B_\pi(X).$$

Order 3:

$$\begin{aligned} E(X^3) &= k^B_{\sqcap\!\sqcap}(X) \ + \ k^B_{|\sqcap}(X) \ + \ k^B_{\sqcap|}(X) \ + \ k^B_{\overline{\,|\,|\,|\,}}(X) + k^B_{|||}(X) \\ &= k^B_3(X) \ + \ k^B_1(X) k^B_2(X) \ + \ k^B_2(X) k^B_1(X) \\ &\quad + \ k^B_2(X k^B_1(X), X) \ + \ k^B_1(X) k^B_1(X) k^B_1(X) \end{aligned}$$

[4] J.A. Mingo, R. Speicher, Free Probability and Random Matrices, Fields Inst. Monographs, Springer New York, 2017.

Multilinear function series

Dykema[5] studied in this setting the counterparts of the R-, S- and T-transforms introducing the notion of *formal multilinear function series*, Mul$[[B]]$, which now play the role of formal power series.

$$(\alpha_n : B^{\otimes n} \to B)_{n \geqslant 0} \qquad B^{\otimes 0} = \mathbb{C}$$

$X \in A$, $E(X) = 1_A$

$$\Phi_X = (\Phi_{X,0}, \Phi_{X,1}, \ldots) \in \mathrm{Mul}[[B]]$$

$\Phi_{X,0} = E(X)$ and for $n > 0$

$$\Phi_{X,n}(b_1, \ldots, b_n) = E(Xb_1Xb_2 \cdots Xb_nX)$$

[5] K. Dykema, *Multilinear function series and transforms in free probability theory*, Adv Math 208 (2007) 351.

$\alpha, \beta \in \mathrm{Mul}[[B]]$, $\beta_0 = 0$: *composition* $\alpha \circ \beta \in \mathrm{Mul}[[B]]$

$$(\alpha \circ \beta)_n(b_1, \ldots, b_n) = \sum_{k=1}^{n} \sum_{\substack{p_1,\ldots,p_k \geqslant 1 \\ p_1+\cdots+p_k=n}} \alpha_k\Big(\beta_{p_1}(b_1, \ldots, b_{p_1}), \ldots, \beta_{p_k}(b_{q_k+1}, \ldots, b_{q_k+p_k})\Big)$$

$$= \sum_{k=1}^{n} \sum_{\substack{p_1,\ldots,p_k \geqslant 1 \\ p_1+\cdots+p_k=n}} \gamma\big(\alpha_k \otimes \beta_{p_1} \otimes \cdots \otimes \beta_{p_k}\big)(b_1, \ldots, b_n)$$

$q_j := p_1 + \cdots + p_{j-1}$.

$$I = (0, \mathrm{id}_B, 0, \ldots).$$

This is the operad End_B of all multilinear maps on B,

$$\mathrm{End}_B(n) = \mathrm{Hom}_{\mathrm{Vect}}(B^{\otimes n}, B),$$

with γ as operadic product.

$\alpha, \beta \in \mathrm{Mul}[[B]]$: $\alpha \cdot \beta \in \mathrm{Mul}[[B]]$, $1 = (1_A, 0, \ldots)$

$$(\alpha \cdot \beta)_n(b_1, \ldots, b_n) = \sum_{k=0}^{n} \alpha_k(b_1, \ldots, b_k)\beta_{n-k}(b_{k+1}, \ldots, b_n)$$
$$= \sum_{k=0}^{n} \gamma(m_B \otimes \alpha_k \otimes \beta_{n-k})(b_1, \ldots, b_n)$$

Note: Dykema showed $(\alpha \cdot \beta) \circ \tau = (\alpha \circ \tau) \cdot (\beta \circ \tau)$

$$G_0 = \{\alpha \in \mathsf{Mul}[[B]] \mid \alpha_0 = 0\} \qquad G_1 = \{\alpha \in \mathsf{Mul}[[B]] \mid \alpha_0 = 1\}$$

$\beta \in \mathrm{Mul}[[B]]$: (unsymmetrized) R-transform of β

$$R_\beta = \left[(1 + \beta \cdot I)^{-1} \cdot \beta\right] \circ (I + I \cdot \beta \cdot I)^{<-1>}$$

General viewpoint: operads with multiplication $(\mathcal{P}, \gamma_{\mathcal{P}}, m)$

Operad with multiplication

Abstract setting of an operad with multiplication $(\mathcal{P}, \gamma_{\mathcal{P}}, m)$, $m \in \mathcal{P}(2)$:

A (non-symmetric) operad $(\mathcal{P}, \gamma_{\mathcal{P}})$ (in category Vect) is a sequence of vector spaces $\mathcal{P}(n)$, $n \in \mathbb{N}$, with a unit operation $I \in \mathcal{P}(1)$ and composition:

$$\begin{aligned}\gamma_{\mathcal{P}} : \mathcal{P}(r) \ \otimes \ (\mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_r)) &\to \mathcal{P}(n_1 + \cdots + n_r) \\ p \otimes (q_1 \otimes \cdots \otimes q_r) &\mapsto \gamma_{\mathcal{P}}(p \otimes (q_1 \otimes \cdots \otimes q_r)),\end{aligned}$$

for $r \in \mathbb{N}$ and $n_1, \ldots, n_r \in \mathbb{N}$, satisfying the axioms

$$\gamma_{\mathcal{P}}(I \otimes q) = q, \qquad \gamma_{\mathcal{P}}(p \otimes (I \otimes \cdots \otimes I)) = p,$$

and $\gamma_{\mathcal{P}}\big(\gamma_{\mathcal{P}}\big(p \otimes (q_1 \otimes \cdots \otimes q_r)\big) \otimes (h^1_1 \otimes \cdots \otimes h^1_{n_1} \otimes \cdots \otimes h^r_1 \otimes \cdots \otimes h^r_{n_r})\big)$

$$= \gamma_{\mathcal{P}}\big(p \otimes \big(\gamma_{\mathcal{P}}(q_1 \otimes (h^1_1 \otimes \cdots \otimes h^1_{n_1})) \otimes \cdots \otimes \gamma_{\mathcal{P}}(q_r \otimes (h^r_1 \otimes \cdots \otimes h^r_{n_r}))\big)\big).$$

Multiplication $m \in \mathcal{P}(2)$:

$$\gamma_{\mathcal{P}}(m \otimes \mathrm{id} \otimes m) = \gamma_{\mathcal{P}}(m \otimes m \otimes \mathrm{id}).$$

Formal series in operators from $\mathcal{P}$

$$\mathbb{C}[[\mathcal{P}]] = \prod_{n>0} P(n)$$

[Chapoton 2002][6] $(\mathbb{C}[[\mathcal{P}]], \times, \mathrm{id})$ is an associative monoid and $\alpha \in \mathbb{C}[[\mathcal{P}]]$ is invertible if and only if $\alpha_1 \neq 0$.

$$G^{\text{diff}} := \{\alpha \in \mathbb{C}[[\mathcal{P}]] \mid \alpha_1 = 1\}.$$

The set G^{diff} with the composition product $\times$ is a group.

[6] F. Chapoton, *Rooted trees and an exponential-like series*, arXiv:math/0209104.

Define a multiplication: $\alpha, \beta \in \mathbb{C}[[\mathcal{P}]]$

$$\alpha \cdot \beta = \sum_{n \geqslant 1} \sum_{\substack{k,q \geqslant 0 \\ k+q=n}} \gamma_{\mathcal{P}}(m \otimes \alpha_k \otimes \beta_q).$$

$$\mathbb{C}[[\mathcal{P}]]_0 := \mathbb{C}1 \oplus \mathbb{C}[[\mathcal{P}]], \ \deg\, 1 = 0$$

$$G^{\mathsf{inv}} := \{x \in \mathbb{C}[[\mathcal{P}]]_0 \mid x_0 = 1\}.$$

The set G^{inv} with the product $\cdot$ is a group.

We have introduced so far two formal groups, G^{diff} and G^{inv}. Generic elements of the groups G^{diff} and G^{inv} will be denoted g respectively h.

Left- and right action, distributivity

Def.: For $h \in G^{\text{inv}}$ and $g \in G^{\text{diff}}$ we define

1) $(G^{\text{inv}} \leftharpoondown G^{\text{diff}})$ $1 \leftharpoondown g := 1$ and

$$h \leftharpoondown g := h \times g$$

2) $(G^{\text{inv}} \rightharpoondown G^{\text{diff}})$

$$h \rightharpoondown g = h \cdot g \cdot h^{-1}.$$

Prop.: For $h, h' \in G^{\text{inv}}$ and $g, g' \in G^{\text{diff}}$, we find:

$$i)\ (h \cdot h') \leftharpoondown g = (h \leftharpoondown g) \cdot (h' \leftharpoondown g), \qquad ii)\ (h \leftharpoondown g)^{-1} = h^{-1} \leftharpoondown g.$$

$$iii)\ (h \rightharpoondown g') \times g = (h \leftharpoondown g) \rightharpoondown (g' \times g)$$

Proof: Thanks to the definition of the product $\cdot$ and the action $\leftharpoondown$ we have i)

$$
\begin{aligned}
(h \cdot h') \leftharpoondown g &= 1 + \sum_{\substack{k,q \geqslant 0 \\ k+q \geqslant 1}} \gamma_{\mathcal{P}}\big(\gamma_{\mathcal{P}}(m \otimes h_k \otimes h'_q) \otimes g \otimes \cdots \otimes g\big) \\
&= 1 + \sum_{\substack{k,q \geqslant 0 \\ k+q \geqslant 1}} \gamma_{\mathcal{P}}\big(m \otimes \gamma_{\mathcal{P}}(h_k \otimes g \otimes \cdots \otimes g) \otimes \gamma_{\mathcal{P}}(h_q \otimes g \otimes \cdots \otimes g)\big) \\
&= (h \leftharpoondown g) \cdot (h' \leftharpoondown g)
\end{aligned}
$$

ii)

$$
\begin{aligned}
(h \rightharpoondown g') \times g &= (h \leftharpoondown g) \cdot (g \times g') \cdot (h^{-1} \leftharpoondown g) \\
&= (h \leftharpoondown g) \cdot (g \times g') \cdot (h \leftharpoondown g)^{-1} \\
&= (h \leftharpoondown g) \rightharpoondown (g' \times g).
\end{aligned}
$$

Left and right translations[7]

$$\begin{array}{rccc} \lambda: & G^{\mathsf{inv}} & \to & G^{\mathsf{diff}}, \\ & h & \mapsto & I \cdot h \end{array} \qquad \begin{array}{rccc} \rho: & G^{\mathsf{inv}} & \to & G^{\mathsf{diff}} \\ & h & \mapsto & h \cdot I \end{array}$$

G^{inv} can be endowed with two additional products: $h, h' \in G^{\mathsf{inv}}$

$$h \star_l h' = h' \cdot (h \leftharpoondown \lambda(h')), \qquad h \star_r h' = (h \leftharpoondown \rho(h')) \cdot h'$$

Prop.: Let $h, h' \in G^{\mathsf{inv}}$ and $g \in G^{\mathsf{diff}}$. Then we have

$$\lambda(h) \times \lambda(h') = \lambda(h \star_l h'), \qquad \rho(h) \times \rho(h') = \rho(h \star_r h')$$

$$\lambda(h) \times \rho(h') = h' \rightharpoondown \lambda(h \star_r h') \qquad \rho(h) = h \rightharpoondown \lambda(h)$$

[7] A. Frabetti, *Groups of tree-expanded series*, J. Algebra 319 (2008) 377.

A factorisation

Def.: Let $r_a, r_b \in G^{\mathsf{inv}}$ and define the free product in G^{inv}

$$r_a \boxtimes r_b := r_b \star_l v_{a,b}$$

where

$$v_{a,b} := r_a \leftharpoondown \rho(r_b \leftharpoondown \lambda(v_{a,b}))$$

Thm.: Let $r_a, r_b \in G^{\mathsf{inv}}$. Suppose for $t_a, t_b \in G^{\mathsf{inv}}$ the following fixed point equations in G^{inv} to hold

$$r_a = t_a \leftharpoondown \lambda(r_a), \qquad r_b = t_b \leftharpoondown \lambda(r_b).$$

Then

$$r_a \boxtimes r_b = \big([t_a \leftharpoondown (t_b \rightharpoondown I)] \cdot t_b \big) \leftharpoondown \lambda(r_a \boxtimes r_b).$$

Dykema's T-transform

Remark: in the case of $\mathsf{Mul}[[B]]$, we obtain the relation between R-transforms and T-transforms, $R_X, R_Y, T_X, T_Y \in \mathsf{Mul}[[B]]$ for $X, Y \in A$

$$R_X = T_X \leftharpoondown \lambda(R_X) \qquad R_Y = T_Y \leftharpoondown \lambda(R_Y)$$

$$\begin{aligned} T_{XY} \leftharpoondown \lambda(R_{XY}) &= \Big(\big(T_X \leftharpoondown (T_Y \rightharpoondown I)\big) \cdot T_Y\Big) \leftharpoondown \lambda(R_{XY}) \\ &= \Big(\big(T_X \leftharpoondown (T_Y \cdot I \cdot T_Y^{-1})\big) \cdot T_Y\Big) \leftharpoondown \lambda(R_{XY}) \\ &= \Big(\big(T_X \circ (T_Y \cdot I \cdot T_Y^{-1})\big) \cdot T_Y\Big) \leftharpoondown \lambda(R_{XY}) \end{aligned}$$

Freeness of X and Y enters through the product $R_{XY} := R_X \boxtimes R_Y$

The product $R_{XY} := R_X \boxtimes R_Y$, seen as an element in $\mathsf{Mul}[[B]]$, and the recursion defining it amounts to computing the following expression for free random variables X, Y and $b_0, \ldots, b_{n-1}$, $c_1, \ldots, c_n \in B$:

$$\begin{aligned} &k_n^B(b_0Xc_1Y, b_1Xc_2Y, \ldots, b_{n-1}Xc_nY) \\ &\quad = \sum_{\pi \in NC_n} k^B_{\pi, \mathsf{Kr}(\pi)}(b_0Xc_1Y, b_1Xc_2Y, \ldots, b_{n-1}Xc_nY) \end{aligned}$$

Putting $c_1 = \cdots = c_n = 1_A$ gives

$$k_n^B(b_0XY, b_1XY, \cdots, b_{n-1}XY) = (R_X \boxtimes R_Y)_n(b_0XY, b_1XY, \ldots, b_{n-1}XY).$$

Proof

Proof: Set $r_{ab} := r_a \boxtimes r_b = r_b \star_l v_{a,b}$. Recall that $(G^{\text{inv}}, \cdot) \leftharpoondown (G^{\text{diff}}, \times)$ and $r_a = t_a \leftharpoondown \lambda(r_a)$, $\quad r_b = t_b \leftharpoondown \lambda(r_b)$,

$$(\star) \quad \lambda(h) \times \lambda(h') = \lambda(h \star_l h')$$

$$\begin{aligned} r_{ab} = r_b \star_l v_{a,b} &:= v_{a,b} \cdot (r_b \leftharpoondown \lambda(v_{a,b})) \\ &= v_{a,b} \cdot \big((t_b \leftharpoondown \lambda(r_b)) \leftharpoondown \lambda(v_{a,b})\big) \\ &= v_{a,b} \cdot \big(t_b \leftharpoondown (\lambda(r_b) \times \lambda(v_{a,b}))\big) \\ &= v_{a,b} \cdot (t_b \leftharpoondown \lambda(r_b \star_l v_{a,b})) \\ &= v_{a,b} \cdot (t_b \leftharpoondown \lambda(r_{ab})) \end{aligned}$$

Remark: We observe that

$$v_{a,b} \cdot (t_b \leftharpoondown \lambda(r_{ab})) = [v_{a,b} \leftharpoondown I] \cdot (t_b \leftharpoondown \lambda(r_{ab}))$$

$$= \left(v_{a,b} \leftharpoondown \left[\lambda(r_{ab})^{\langle -1 \rangle} \times \lambda(r_{ab})\right]\right) \cdot (t_b \leftharpoondown \lambda(r_{ab}))$$
$$= \left(\left([v_{a,b} \leftharpoondown \lambda(r_{ab})^{\langle -1 \rangle}\right] \leftharpoondown \lambda(r_{ab})\right) \cdot (t_b \leftharpoondown \lambda(r_{ab}))$$
$$= \left(\left[v_{a,b} \leftharpoondown \lambda(r_{ab})^{\langle -1 \rangle}\right] \cdot t_b\right) \leftharpoondown \lambda(r_{ab})$$

Hence, in principle, we need to compute/show that

$$v_{a,b} \leftharpoondown \lambda(r_{ab})^{\langle -1 \rangle} \stackrel{?}{=} [t_a \leftharpoondown (t_b \rightharpoondown I)] \cdot t_b$$

Now we use the fixed point equations $r_a = t_a \leftharpoondown \lambda(r_a)$, $r_b = t_b \leftharpoondown \lambda(r_b)$ in the fixed point equation for $v_{a,b}$,

$$v_{a,b} = r_a \leftharpoondown \rho(r_b \leftharpoondown \lambda(v_{a,b}))$$
$$= [t_a \leftharpoondown \lambda(r_a)] \leftharpoondown \rho(r_b \leftharpoondown \lambda(v_{a,b}))$$
$$= t_a \leftharpoondown \left[\lambda(r_a) \times \rho(r_b \leftharpoondown \lambda(v_{a,b}))\right]$$

$$= t_a \leftharpoondown \big[\lambda(r_a) \times \rho[t_b \leftharpoondown (\lambda(r_b) \times \lambda(v_{a,b}))]\big]$$
$$= t_a \leftharpoondown \big[\lambda(r_a) \times \rho(t_b \leftharpoondown \lambda(r_b \star_l v_{a,b}))\big]$$
$$= t_a \leftharpoondown \big[\lambda(r_a) \times \rho(t_b \leftharpoondown \lambda(r_{ab}))\big].$$

Next we use the 'mixed' identity

$$(\star\star) \quad \lambda(h) \times \rho(h') = (I \cdot h) \times (h' \cdot I) = h' \rightharpoondown \lambda(h \star_r h')$$

with $h' := t_b \leftharpoondown \lambda(r_{ab})$ and $h := r_a$. This then yields

$$\lambda(r_a) \times \rho(t_b \leftharpoondown \lambda(r_{ab})) = \big(t_b \leftharpoondown \lambda(r_{ab})\big) \rightharpoondown \lambda\big(r_a \star_r (t_b \leftharpoondown \lambda(r_{ab}))\big).$$

Inserting the fixed point equation for r_b into $v_{a,b}$, we get also

$$v_{a,b} = r_a \leftharpoondown \rho\big(t_b \leftharpoondown \lambda(r_b \star_l v_{a,b})\big) = r_a \leftharpoondown \rho(t_b \leftharpoondown \lambda(r_{ab}))$$

$$= \left[r_a \leftharpoonup \rho(t_b \leftharpoonup \lambda(r_{ab}))\right] \cdot (t_b \leftharpoonup \lambda(r_{ab})) \cdot (t_b \leftharpoonup \lambda(r_{ab}))^{-1}$$
$$= r_a \star_r (t_b \leftharpoonup \lambda(r_{ab})) \cdot (t_b \leftharpoonup \lambda(r_{ab}))^{-1}.$$

Hence, $v_{a,b} \cdot (t_b \leftharpoonup \lambda(r_{ab})) = r_a \star_r (t_b \leftharpoonup \lambda(r_{ab}))$ implies

$$\lambda(r_a) \times \rho(t_b \leftharpoonup \lambda(r_{ab})) = [t_b \leftharpoonup \lambda(r_{ab})] \rightharpoonup \lambda(v_{a,b} \cdot (t_b \leftharpoonup \lambda(r_{ab})))$$
$$= (t_b \leftharpoonup \lambda(r_{ab})) \rightharpoonup \lambda(r_{ab}).$$

We obtain for $v_{a,b}$ and r_{ab},

$$v_{a,b} = t_a \leftharpoonup \Big((t_b \leftharpoonup \lambda(r_{ab})) \rightharpoonup \lambda(r_{ab}) \Big)$$

It follows then from

$$(\star\star\star) \quad (h \leftharpoonup g) \rightharpoonup (g' \times g) = (h \rightharpoonup g') \times g$$

with $h = t_b$, $g := \lambda(r_{ab})$ and $g' := I$

$$v_{ab} = t_a \leftharpoondown \big((t_b \rightharpoonup I) \times \lambda(r_{ab})\big)$$
$$= \big(t_a \leftharpoondown (t_b \rightharpoonup I)\big) \leftharpoondown \lambda(r_{ab}).$$

It follows then

$$r_{ab} = v_{a,b} \cdot \big[t_b \leftharpoondown \lambda(r_{ab})\big]$$
$$= \Big[\big(t_a \leftharpoondown (t_b \rightharpoonup I)\big) \leftharpoondown \lambda(r_{ab})\Big] \cdot \Big[t_b \leftharpoondown \lambda(r_{ab})\Big]$$
$$= \Big(\big(t_a \leftharpoondown (t_b \rightharpoonup I)\big) \cdot t_b\Big) \leftharpoondown \lambda(r_{ab})$$
$$= \Big(\big[t_a \leftharpoondown (t_b \cdot I \cdot t_b^{-1})\big] \cdot t_b\Big) \leftharpoondown \lambda(r_{ab}).$$

Double product

$$t_a \blacksquare t_b := t_{ab} := (t_a \leftharpoonup (t_b \rightharpoonup I)) \cdot t_b$$

Observe that

$$
\begin{aligned}
(t_a \blacksquare t_b) \blacksquare t_c &= t_{(ab)c} \\
&= \big((t_a \leftharpoonup (t_b \rightharpoonup I)) \cdot t_b) \leftharpoonup (t_c \rightharpoonup I)\big) \cdot t_c \\
&= \big(((t_a \leftharpoonup (t_b \rightharpoonup I)) \leftharpoonup (t_c \rightharpoonup I)) \cdot (t_b \leftharpoonup (t_c \rightharpoonup I))\big) \cdot t_c \\
&= \big((t_a \leftharpoonup ((t_b \rightharpoonup I) \times (t_c \rightharpoonup I))) \cdot (t_b \leftharpoonup ((t_c \rightharpoonup I))\big) \cdot t_c \\
&= \big((t_a \leftharpoonup ((t_b \cdot I \cdot t_b^{-1}) \times (t_c \rightharpoonup I))) \cdot (t_b \leftharpoonup (t_c \rightharpoonup I))\big) \cdot t_c \\
&= \Big(t_a \leftharpoonup \big(t_b \leftharpoonup (t_c \rightharpoonup I) \cdot t_c \cdot I \cdot t_c^{-1} \cdot (t_b \leftharpoonup (t_c \rightharpoonup I))^{-1}\big)\Big) \cdot (t_b \leftharpoonup (t_c \rightharpoonup I)) \cdot t_c \\
&= \Big(t_a \leftharpoonup \big(t_b \leftharpoonup (t_c \rightharpoonup I) \cdot t_c \cdot I \cdot (t_b \leftharpoonup (t_c \rightharpoonup I) \cdot t_c)^{-1}\big)\Big) \cdot (t_b \leftharpoonup (t_c \rightharpoonup I)) \cdot t_c \\
&= t_{a(bc)} = t_a \blacksquare (t_b \blacksquare t_c)
\end{aligned}
$$

Associative product

The last computation implies for

$$r_{(ab)c} := (r_a \boxtimes r_b) \boxtimes r_c \qquad r_{a(bc)} := r_a \boxtimes (r_b \boxtimes r_c)$$

$$t_{a(bc)} \leftharpoondown \lambda(r_{a(bc)}) = r_{a(bc)} = t_{(ab)c} \leftharpoondown \lambda(r_{a(bc)})$$

and

$$t_{(ab)c} \leftharpoondown \lambda(r_{(ab)c}) = r_{(ab)c}$$

Hence, $r_{(ab)c}$ and $r_{a(bc)}$ solve the same fixed point equation and therefore

$$r_{(ab)c} = r_{a(bc)}.$$

Invertibility

$$t_a \blacksquare 1 = 1 \blacksquare t_a = 1.$$

$$u \blacksquare t_a = t_a \blacksquare u = 1$$
$$u \blacksquare t_a = 1 \implies u = t_a^{-1} \curvearrowleft [t_a \curvearrowright I]^{\langle -1 \rangle}$$

$$\begin{aligned} t_a \blacksquare u &= \big(t_a \curvearrowleft [u \curvearrowright I]\big) \cdot u \\ &= \big(t_a \curvearrowleft \big[[t_a^{-1} \curvearrowleft [t_a \curvearrowright I]^{\langle -1 \rangle}] \curvearrowright I\big]\big) \cdot u \\ &= \Big(t_a \curvearrowleft \big(\big[t_a^{-1} \curvearrowleft [t_a \curvearrowright I]^{\langle -1 \rangle}\big] \curvearrowright I\big)\Big) \cdot \big(t_a^{-1} \curvearrowleft [t_a \curvearrowright I]^{\langle -1 \rangle}\big) \end{aligned}$$

We use the recursion

$$X = [t_a^{-1} \curvearrowleft X] \curvearrowright I$$

such that

$$[t_a \curvearrowright I]^{\langle -1 \rangle} = \big(t_a^{-1} \curvearrowleft [t_a \curvearrowright I]^{\langle -1 \rangle}\big) \curvearrowright I$$

Indeed:

$$\begin{aligned}
\big([t_a^{-1} \curvearrowleft [t_a \curvearrowright I]^{\langle -1 \rangle}] \curvearrowright I\big) \times [t_a \curvearrowright I] \\
&= \Big(\big[t_a^{-1} \curvearrowleft [t_a \curvearrowright I]^{\langle -1 \rangle}\big] \curvearrowleft [t_a \curvearrowright I]\Big) \curvearrowright [t_a \curvearrowright I] \\
&= \Big(t_a^{-1} \curvearrowleft \big[[t_a \curvearrowright I]^{\langle -1 \rangle} \times [t_a \curvearrowright I]\big]\Big) \curvearrowright [t_a \curvearrowright I] \\
&= [t_a^{-1} \curvearrowleft I] \curvearrowright [t_a \curvearrowright I] \\
&= t_a^{-1} \curvearrowright [t_a \curvearrowright I] \\
&= t_a^{-1} \cdot \big(t_a \cdot I \cdot t_a^{-1}\big) \cdot t_a \\
&= I
\end{aligned}$$

Further work: Operads of Rooted planar binary trees

From an operad with multiplication, we build $(\lambda, \rho, \curvearrowright, \star_l, \star_r)$ satisfying the relations

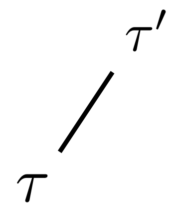
$$
\begin{array}{ll}
\text{(Cocycle)}(\star) & \lambda(h) \times \lambda(h') = \lambda(h \star_l h') \\
(\star\star) & \lambda(h) \times \rho(h') = h' \curvearrowright \lambda(h \star_r h') \\
\text{(Associativity)}(\star\star\star) & (h \curvearrowleft g) \curvearrowright (g' \times g) = (h \curvearrowright g') \times g
\end{array}
$$

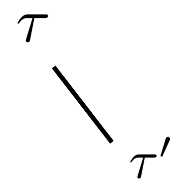
The quintuple $(\lambda, \rho, \curvearrowright, \star_l, \star_r)$ yields $\boxtimes$ and $\blacksquare$ (two groups laws) on G^{inv} + *twisted factorization* (of an abstract T transform, depending on λ).

... are there other ways to produce such a quintuple $(\lambda, \rho, \curvearrowright, \star_l, \star_r)$ satisfying $(\star)$-$(\star\star\star)$?

$\mathbb{C}[[\mathcal{P}]] \simeq$ Rooted (planar) binary trees

Frabetti introduces two binary operations on planar binary trees: the Over operation $/$ and the Under $\backslash$

$\tau\backslash\tau' =$ 

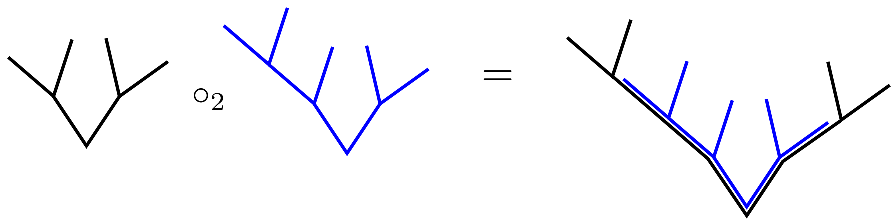
$\tau/\tau' =$ 

and considers the following left and right translations:

$$\lambda(t) := \curlyvee \backslash t, \quad \rho(t) := t / \curlyvee$$

How can we complete the pair (λ, ρ) to a good quintuple $(\lambda, \rho, \rightharpoonup, \star_l, \star_r)$?

OverUnder Operad (Duplicial Operad)



Both $/$ and $\backslash$ are multiplications and

$$(\cdot/(\cdot\backslash\cdot)) = ((\cdot/\cdot)\backslash\cdot))$$

($\implies$ $\boxtimes_{\mathsf{O}}$ and $\boxtimes_{\mathsf{U}}$).

Define

$$t \curvearrowright g := t / g \backslash t^{\backslash -1}, \quad t \star_l t' := t' \backslash (t \curvearrowleft \lambda(t'))$$

$$r_a \boxtimes_{\mathsf{OU}} r_b := v_{a,b} \backslash r_b \curvearrowleft_{\mathsf{OU}} v_{a,b}$$

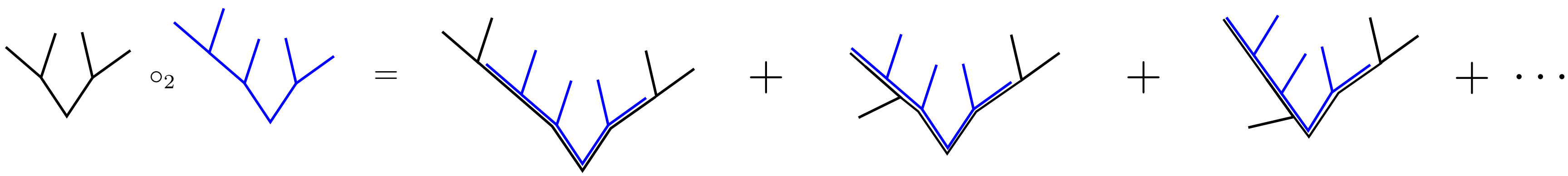
The following twisted factorization holds:

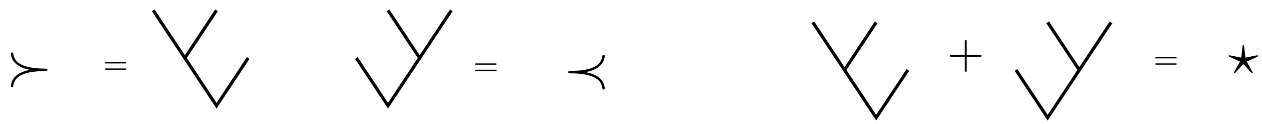
$$t(r_a \boxtimes_{\mathsf{OU}} r_b) = t(r_a) \curvearrowleft_{\mathsf{OU}} (t(r_b) \curvearrowright \mathsf{Y}) \backslash t(r_b) := t(r_a) \blacksquare_{\mathsf{OU}} t(r_b)$$

$$r := t(r) \curvearrowleft_{\mathsf{OU}} \lambda(r)$$

But $\boxtimes_{\mathsf{OU}}, \blacksquare_{\mathsf{OU}.}$ are merely **magmatic** operations... (not associative)

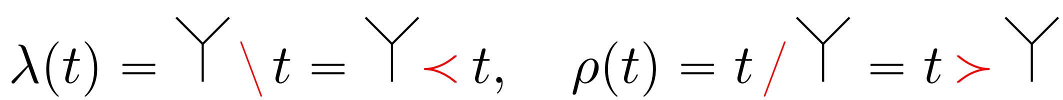
Dendriform operad





$$((\cdot \prec \cdot) \prec \cdot) = (\cdot \prec (\cdot \star \cdot)), \quad (\cdot \succ (\cdot \succ \cdot)) = ((\cdot \star \cdot) \succ \cdot)$$

$$(\cdot \succ (\cdot \prec \cdot)) = ((\cdot \succ \cdot) \prec \cdot)$$



Define

$$t \curvearrowright g := t \succ g \prec t^{\star -1},$$

$$t \star_l t' := t' \star (t \curvearrowleft_{\text{Dend.}} \lambda(t')), \quad t \star_r t' := (t \curvearrowleft_{\text{Dend.}} \rho(t')) \star t'.$$

$$r_a \boxtimes_{\text{Dend.}} r_b := v_{a,b} \star \left[r_b \curvearrowleft_{\text{Dend.}} \lambda(v_{a,b}) \right]$$

The following twisted factorization holds:

$$t(r_a \boxtimes_{\text{Dend.}} r_b) = \left[t(r_a) \curvearrowleft_{\text{Dend.}} (t(r_b) \curvearrowright \curlyvee) \right] \star t(r_b) := t(r_a) \blacksquare_{\text{Dend.}} t(r_b)$$

$$r := t(r) \curvearrowleft_{\text{Dend.}} \lambda(r)$$

Each binary operation $\boxtimes_{\text{Dend.}}$ and $\blacksquare_{\text{Dend.}}$ induces a **group** product on G^{inv}...

$$
\begin{aligned}
t \curvearrowright (t' \curvearrowright g) &= (t' \succ ((t \succ g) \prec t^{-1})) \prec t'^{-1} \\
&= ((t' \succ (t \succ g)) \prec t^{-1}) \prec t'^{-1} \\
&= (((t' \star t) \succ g) \prec t^{-1}) \prec t'^{-1} \\
&= (t' \star t) \succ g \prec (t' \star t)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\lambda(t) \times \rho(t') &= (\curlyvee \prec t) \times (t' \succ \curlyvee) \\
&= (t' \succ \curlyvee) \prec (t \times (t' \succ \curlyvee)) \\
&= t' \succ (\curlyvee \prec (t \times (t' \succ \curlyvee))) \\
&= t' \succ ((\curlyvee \prec (t \times (t' \succ \curlyvee))) \prec t' \star t'^{\star -1}) \\
&= t' \succ (((\curlyvee \prec (t \times (t' \succ \curlyvee))) \prec t') \prec t'^{\star -1}) \\
&= t' \succ ((\curlyvee \prec ((t \times (t' \succ \curlyvee)) \star t')) \prec t'^{\star -1}) \\
&= t' \curvearrowright \lambda(t \star_r t')
\end{aligned}
$$

cointeracting coalgebras & n.c probability

Multiplicative functions on the poset of non-crossing partitions
$\to$ Character on a co-dendriform bi-algebra $(B_{\text{gap.}}, \Delta^{\prec}_{\text{gap}}, \Delta^{\succ}_{\text{gap.}})$
$\to$ $B_{\text{gap.}}$ 'Incidence bialgebra' of the *gap-insertion operad.*

$$(B^{\star}_{\text{gap}}, \prec, \succ)$$

To Mbius inversion corresponds an operad, the *refinement operad*.
$\to$ 'Incidence bialgebra' $B_{\text{ref.}}, \Delta_{\text{ref.}}$

$$B^{\star}_{\text{gap.}} \leftharpoondown \text{Spec}(B_{\text{ref.}})$$

$$(h \prec h') \leftharpoondown g = (h \leftharpoondown g) \prec (h' \leftharpoondown g),$$
$$(h \succ h') \leftharpoondown g = (h \leftharpoondown g) \succ (h' \leftharpoondown g).$$

Thank you for your attention!