

On Random Permutations Sampled by Surface Groups

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1. Introduction and Main Results

S_N = Symmetric group on N elements

$\Gamma_g = \text{Th}(\text{genus } g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$

For simplicity: $g=2$

$\Gamma = \Gamma_2 = \text{Th}(\text{genus } 2) = \langle a, b, c, d \mid [a, b][c, d] = 1 \rangle$

We study $\text{Hom}(\Gamma, S_N) = \text{set of group homomo.}$

$$= \{(\alpha, \beta, \gamma, \delta) \in S_N^4 \mid [\alpha, \beta][\gamma, \delta] = 1\}$$

Q: $|\text{Hom}(\Gamma, S_N)| = ?$

For a modern proof: see appendix by Zagier to the book "Graphs on Surfaces and their Applications" by Lando and Zvonkin.

Frobenius + Hurwitz : $|\text{Hom}(\Gamma, G)| = |G|^3 \cdot \sum_{\rho \in \text{Irrep}(G)} \frac{1}{(\dim \rho)^2}$

1896 1902

Lubotzky (1996): for S_N $\sum \frac{1}{(\dim \rho)^2} \xrightarrow[N \rightarrow \infty]{} 2$

$$\Rightarrow |\text{Hom}(\Gamma, S_N)| \approx 2(N!)^3$$

Our question: Fix $\tau \in \Gamma$. (for example $\begin{pmatrix} a \\ a, b \end{pmatrix}$)

τ induces a probability distribution on S_N :
 pick $\phi \in \text{Hom}(\Gamma, S_N)$ unif. at random,
 and evaluate $\phi(\tau)$.

$\Gamma \leftarrow \{ \gamma \in \text{Isom}(\mathbb{H}^n, \mathbb{H}^n) \mid \dots \}$

and evaluate $\phi(\gamma)$.

Goal: (1) Understand this random permutation $\phi(\gamma)$
(2) In particular: what is $E[\#\text{fixed points}]$ of $\phi(\gamma)$

Main Result: fix $\gamma \in \Gamma$. $E[\#\text{fix } \phi(\gamma)]$

Thm A: There is an infinite sequence of rational numbers $a_{-1}(\gamma), a_0(\gamma), a_1(\gamma), \dots \in \mathbb{Q}$ such that for every $N \in \mathbb{Z}_{\geq 0}$

$$E[\#\text{fix } \phi(\gamma)] = a_{-1}(\gamma) \cdot N + a_0(\gamma) + \frac{a_1(\gamma)}{N} + \dots + \frac{a_n(\gamma)}{N^n} + O\left(\frac{1}{N^{n+1}}\right)$$

Example: $\gamma = a$:

$$1 + \frac{1}{N^2} + \frac{2}{N^3} + \frac{4}{N^4} + \dots$$

Thm B: If $\gamma \neq \text{id}$ write $\gamma = \gamma_0^q$ with $q \in \mathbb{Z}_{\geq 0}$ maximal. Then

$$a_{-1}(\gamma) = 0 \quad a_0(\gamma) = d(q) \quad \# \text{divisors of } q$$

Namely: $E[\#\text{fix } \phi(\gamma)] \xrightarrow[N \rightarrow \infty]{} d(q)$

In particular $E[\] \rightarrow 1$ if γ_0 is non-power.

Remark Nica 1994: This is true when Γ is replaced with a free group.

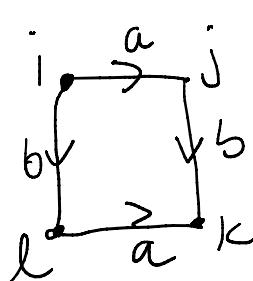
$\Gamma \leftarrow \{ -I, t_0, \dots, r^*(r) \}$ defined by

"The states" on $C^*(\Gamma)$ defined by

$I_n: \Gamma \rightarrow \mathbb{R}$ $I_n(\sigma) = \frac{1}{n} \operatorname{tr} \phi(\sigma)$ converge, as
 $n \rightarrow \infty$, to I_{reg} defined by $I_{\text{reg}}(r) = 0 \forall r \neq 1$.

② Ideas in the proof of Thm A

Consider $\gamma = [a, b] = aba^{-1}b^{-1}$



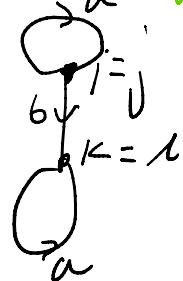
$$i, j, k, l \in \{1, \dots, N\}$$

$$\underline{I_n F_4 = \langle a, b, c, d \rangle}$$

Case 1: i, j, k, l all distinct

$$\frac{\frac{n(n-1)(n-2)(n-3)}{n(n-1) \cdot n(n-1)}}{+} \approx N^{v-e} = N^x$$

Case 2: $i=j \neq k=l$



$$\frac{\frac{n(n-1)}{n(n-1) \cdot n}}{+} \approx N^{v-e} = N^x$$

Case 3:

Euler char
of the
graph

$$\underline{E[\text{fix } [\alpha, \beta]]} = \left(1 + \frac{1}{N-1}\right)^{-1}$$

Surface groups: (1) For every graph and every set of irreducible representations of S_N , we get a contribution which is in fact, an analytic function in N .

an analytic function in N .

Using: - Vershik-Okunkov (1996)

- ideas coming from Weingarten Calculus.

(2) Show that for each graph, only finitely many "sets" of representations contribute more than $\approx \frac{1}{n^m}$.

To illustrate: $\sum_{\rho \in \text{Irreg}(n)} \frac{1}{(\dim_{\rho})^2}$

ρ	dim	$\sum_{\rho \in \text{Irreg}(n)} \frac{1}{(\dim_{\rho})^2}$
sgn	1	$\frac{1}{1^2} = 1$
triv	1	$\frac{1}{1^2} = 1$
std	$n-1$	$\frac{1}{(n-1)^2}$
std \otimes sgn	$n-1$	$\frac{1}{(n-1)^2}$
$\boxed{\text{all others}}$	$\frac{n(n-3)}{2}$	$\approx \frac{4}{n^2}$

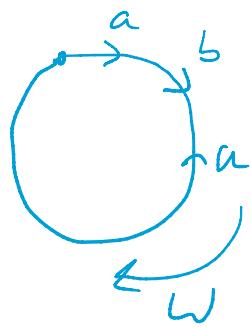
Liebeck-Shalev (04):

Gamburd (06):

$$\sum_{\rho \in \text{Irreg}(n)} \frac{1}{(\dim_{\rho})^2} = \sum_{\text{finitely many irreps}} \frac{1}{(\dim_{\rho})^2} + O\left(\frac{1}{n^m}\right)$$

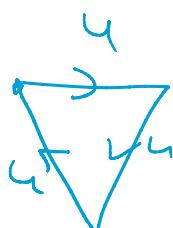
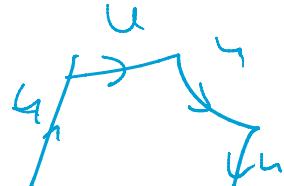
(3) Ideas from Thm B

Free groups: The only graphs with $K=0$ are cycles.

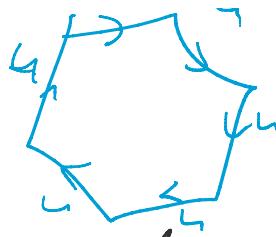


w

$w = u^6$



$\leftarrow \omega$



Surface groups: Contributions of some graph + set of irreps can be of order N^2, N^3, \dots

- Even for graphs you may get wild contrib.

Solution: consider "pieces" of surfaces instead of graphs.

$\text{Hom}(P, S_N) \longleftrightarrow$ N -covering spaces of

We look for 2-dimensional "pieces" of surfaces which contain a loop corresponding to $[a, b]$, and which are embedded in a random covering of

Thm: ① If such a piece γ has nice properties (its boundary is short enough).

then it's contribution to $\oint [\# \text{fix } \phi(z)]$ is $N^{x(\gamma)} + O(N^{x(\gamma)-1})$

② $x(\gamma) \leq 0$

$$\Leftrightarrow \lambda(\gamma) \leq 0$$

③ $\chi(\gamma) < 0 \Leftrightarrow \gamma$ corresponds to a root of \mathfrak{g} .