Subalgebras of Simple AF-Algebras

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AF-embedding problem

Definition (Bratteli '70)

A separable C*-algebra A is approximately finite-dimensional (AF) if there is an increasing sequence $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \subseteq A$ of finite-dimensional C*-algebras such that $\bigcup F_n$ is dense in A.

Question (Effros '82)

1. Is there an abstract characterization of AF-algebras?
2. Is there an abstract characterization of subalgebras of AF-algebras?

Note: A subalgebra of an AF-algebra is typically not an AF-algebra!

For a compact metrizable space X, there is an AF-embedding

$$C(X) \hookrightarrow C(2^{\mathbb{N}}),$$

but $C(X)$ is AF if and only if $\dim(X) = 0$.

Characterizations of AF-algebras

Theorem (Winter et. al. '10)

A separable C*-algebra is an AF-algebra if and only if nuclear dimension zero; i.e. there are $A \xrightarrow[\text{c.p.c.}]{\phi_n} F_n \xrightarrow[\text{c.p.c.}]{\psi_n} A$ such that

- $\|\psi_n(\phi_n(a)) - a\| \to 0$ for all $a \in A$, and
- ψ_n have order zero (preserve orthogonality).

Theorem (consequence of classification)

A simple, infinite-dimensional C*-algebra is AF if and only if

- A is separable, simple, nuclear, $\mathcal{Z}$-stable, and satisfies UCT,
- $T(A) \neq \emptyset$,
- $K_0(A)$ is torsion free,
- $K_0(A) \to \mathrm{Aff}\, T(A)$ has dense range, and
- $K_1(A) = 0$.

Characterization of AF-emeddability

Conjecture (Blackadar-Kirchberg '97)

$A \subseteq \mathcal{B}(H)$ embeds into an AF-algebra if and only if A is exact and quasidiagonal (q.d.); equivalently, there are $A \xrightarrow[\text{c.p.c.}]{\phi_n} F_n \xrightarrow[\text{c.p.c.}]{\psi_n} \mathcal{B}(H)$ such that

- $\|\psi_n(\phi_n(a)) - a\| \to 0$ for all $a \in A$, and
- $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \to 0$ for all $a, b \in A$.

Theorem (S. '18)

A C^*-algebra A satisfying the UCT embeds into a simple unital AF-algebra if and only if A is exact and has a faithful, q.d. trace.

In the context of the conjecture, the extra requirement is:
there are a faithful trace $\tau \in T(A)$ and traces $\tau_n \in T(F_n)$ such that

- $\tau_n(\phi_n(a)) \to \tau(a)$ for all $a \in A$.

THE QUASIDIAGONALITY THEOREM

THEOREM (S. '18)

A C*-algebra A satisfying the UCT embeds into a simple unital AF-algebra if and only if A is exact and has a faithful, q.d. trace.

THEOREM (TIKUISIS-WHITE-WINTER '15, GABE '15)

Suppose A is a separable, exact C*-algebra satisfying the UCT with a faithful trace τ_A.

- τ_A is quasidiagonal if and only if $\pi_{\tau_A}(A)'' \subseteq \mathcal{B}\big(L^2(A,\tau)\big)$ is injective.
- If A is nuclear, τ_A is quasidiagonal.

Some examples

Corollary

If G is a countable, discrete group, TFAE:

- G is amenable;
- $L(G)$ embeds into $\mathcal{R} := \overline{\bigotimes}_{n\geq 1} M_n(\mathbb{C})$;
- $\mathrm{C}^*(G)$ embeds into $\mathcal{Q} := \bigotimes_{n\geq 1} M_n(\mathbb{C})$

Corollary (Lin '08 when $G = \mathbb{Z}^n$)

For G a countable group, X a compact metrizable space, and $G \curvearrowright X$, TFAE:

- $C(X) \rtimes G$ embeds into a simple, unital AF-algebra;
- G is amenable and X admits faithful, G-invariant measure.

The rotation algebras

Consider $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and

$$A_\theta = C(S^1) \rtimes_\theta \mathbb{Z} \cong \mathrm{C}^*(u, v : u, v \text{ unitary}, uv = e^{2\pi i\theta} vu).$$

Theorem (Pimsner-Voiculescu '80)

There is an AF-algebra B_θ and an embedding $A_\theta \hookrightarrow B_\theta$. Further, can arrange

$$K_0(B_\theta) \cong K_0(A_\theta) \cong \mathbb{Z} + \mathbb{Z}\theta \subseteq \mathbb{R}.$$

Remark: It is easy to produce approximate AF-embeddings of A_θ:

That is, there are u.c.p. $\phi_n \colon A_\theta \to M_{q(n)}(\mathbb{C})$ such that

- $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \to 0$ for all $a, b \in A_\theta$, and
- $\|\phi_n(a)\| \to \|a\|$ for all $a \in A_\theta$.

Approximate Embeddings

Write $\theta = \lim_{n\to\infty} \dfrac{p(n)}{q(n)}$. For $n \geq 1$, consider $u_n, v_n \in M_{q(n)}(\mathbb{C})$ given by

$$u_n = \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{bmatrix}, \qquad v_n = \begin{bmatrix} \zeta_n & & & & \\ & \zeta_n^2 & & & \\ & & \zeta_n^3 & & \\ & & & \ddots & \\ & & & & \zeta_n^n \end{bmatrix}.$$

where $\zeta_n = e^{2\pi i p(n)/q(n)}$. Then for all $n \geq 1$,

$$u_n v_n = \zeta_n v_n u_n \approx e^{2\pi i\theta} v_n u_n.$$

Then one can show there are u.c.p. $\phi_n \colon A_\theta \to M_{q(n)}(\mathbb{C})$ with

$$\|\phi_n(u) - u_n\| \to 0 \qquad \text{and} \qquad \|\phi_n(v) - v_n\| \to 0.$$

Approximate Embeddings

This gives approximate embeddings $\phi_n \colon A_\theta \overset{\approx}{\hookrightarrow} M_{q(n)}(\mathbb{C}) \subseteq \mathcal{Q}$.

If one could arrange $(\phi_n)_{n=1}^{\infty}$ to be Cauchy, would have embedding $\phi \colon A_\theta \hookrightarrow \mathcal{Q}$.

Note: No such embedding exists!

The actual proof constructs embeddings

$$M_{q(n-1)}(\mathbb{C}) \oplus M_{q(n)}(\mathbb{C}) \hookrightarrow M_{q(n)}(\mathbb{C}) \oplus M_{q(n+1)}(\mathbb{C})$$

so that the sequence

$$\phi_n \oplus \phi_{n+1} \colon A_\theta \to B_\theta := \varinjlim \left(M_{q(n)}(\mathbb{C}) \oplus M_{q(n+1)}(\mathbb{C}) \right)$$

is Cauchy and then obtains $\phi \colon A_\theta \to B_\theta$.

Approximate AF-Embeddings in general

Theorem (S. '18)

Suppose

- A separable, unital, exact, UCT with faithful qd trace τ_A, and
- B simple, unital, $\mathcal{Q}$-stable AF-algebra with unique trace τ_B.

Suppose $\alpha\colon K_0(A) \to K_0(B)$ is unital and trace-preserving.

- $(\approx \exists)$: There are u.c.p. $\phi_n\colon A \overset{\approx}{\hookrightarrow} B$ with

$$K_0(\phi_n) \approx \alpha \quad \text{and} \quad \tau_B\phi_n \approx \tau_A.$$

- $(\approx !)$: Given $\psi_n\colon A \overset{\approx}{\hookrightarrow} B$ as above, there are $u_n \in U(B)$ with

$$\|u_n\phi_n(a)u_n^* - \psi_n(a)\| \to 0 \qquad \text{for all } a \in A.$$

Note: Given A and τ_A, there always is a B and α: take

$$K_0(B) \cong \operatorname{span}_{\mathbb{Q}} \tau_A(K_0(A)) \subseteq \mathbb{R}.$$

A Restatement

For a C*-algebra B, let

$$B_\omega = \ell^\infty(B)/\{b \in \ell^\infty(B) : \lim_{n\to\omega} \|b_n\| = 0.\}$$

Theorem (S. '18)

Suppose

- A separable, unital, exact, UCT with faithful qd trace τ_A, and
- B simple, unital, $\mathcal{Q}$-stable AF-algebra with unique trace τ_B.

Suppose $\alpha\colon K_0(A) \to K_0(B_\omega)$ is unital and trace-preserving.

- There is nuclear $\phi\colon A \hookrightarrow B_\omega$ with $K_0(\phi) = \alpha$, $\tau_{B_\omega}\phi = \tau_A$.
- Given nuclear $\psi\colon A \hookrightarrow B_\omega$ as above, $\phi \sim_u \psi$.

Remark: The same holds with B in place of B_ω (and $\approx_u$ in place of $\sim_u$).

Obtaining approximate classification

Suppose

- A separable, unital, exact, UCT with faithful qd trace τ_A, and
- B simple, unital, $\mathcal{Q}$-stable AF-algebra with unique trace τ_B.

Quasidiagonality of τ_A provides an trace-preserving approximate embedding

$$\phi_n \colon A \overset{\approx}{\hookrightarrow} M_{q(n)} \subseteq \mathcal{Q}.$$

This gives a nuclear trace-preserving embedding

$$\phi \colon A \hookrightarrow \mathcal{Q}_\omega \hookrightarrow (B \otimes \mathcal{Q})_\omega \cong B_\omega.$$

Goal

Describe all nuclear, trace-preserving $A \hookrightarrow B_\omega$ up to unitary equivalence.

On the uniqueness result

View $B \subseteq \mathcal{B}\big(L^2(B, \tau_B)\big)$ and note that $B'' \cong \mathcal{R}$.

The inclusion $B \hookrightarrow \mathcal{R}$ induces

$$0 \longrightarrow J_B \longrightarrow B_\omega \xrightarrow{q_B} \mathcal{R}^\omega \longrightarrow 0.$$

where

$$J_B = \ker(q_B) \cong \frac{\left\{b \in \ell^\infty(B) : b_n \xrightarrow{SOT} 0\right\}}{\left\{b \in \ell^\infty(B) : b_n \xrightarrow{\|\cdot\|} 0\right\}}.$$

The conditions on (A, τ) for $\pi_\tau(A)'' \subseteq \mathcal{B}\big(L^2(A, \tau_A)\big)$ to be AFD, so any two trace-preserving $A \hookrightarrow \mathcal{R}^\omega$ are unitarily equivalent.

Question

To what extent can we lift a unitary equivalence along q_B?

Lifting unitary equivalence

Consider $\phi, \psi \colon A \to B_\omega$ trace-preserving, nuclear.

$$
\begin{array}{ccccccccc}
 & & & & A & & & & \\
 & & & & \psi \downarrow\downarrow \phi & & & & \\
0 & \longrightarrow & J_B & \longrightarrow & B_\omega & \xrightarrow{q_B} & \mathcal{R}^\omega & \longrightarrow & 0 \\
 & & \| & & \cap\!\!| & & \cap\!\!| & & \\
0 & \longrightarrow & J_B & \longrightarrow & M(J_B) & \xrightarrow{\bar{q}_B} & Q(J_B) & \longrightarrow & 0
\end{array}
$$

As $q_B\phi \sim_u q_B\psi$, there is an $F \in B_\omega \subseteq M(J_B)$ such that

$$F^*F - 1 \in J_B, \quad FF^* - 1 \in J_B, \quad \text{and} \quad F\phi(a) - \psi(a)F \in J_B$$

for all $a \in A$.

This defines $[\phi, \psi, F] \in KK^0_{\mathrm{nuc}}(A, J_B)$. (This is independent of F.)

Obtaining uniqueness

Using UCT (and K-theory of J_B),

$$[\phi, \psi, F] = 0 \in KK^0_{\mathrm{nuc}}(A, J_B) \iff K_0(\phi) = K_0(\psi).$$

In this case, have $\pi : A \to M(J_B \otimes \mathcal{K})$ and $U \in U(M(J_B \otimes \mathcal{K})$ with

$$\begin{pmatrix} F & 0 \\ 0 & 1 \end{pmatrix} - U \in J_B \otimes \mathcal{K} \quad \text{and} \quad U \begin{pmatrix} \phi(a) & 0 \\ 0 & \pi(a) \end{pmatrix} U^* = \begin{pmatrix} \psi(a) & 0 \\ 0 & \pi(a) \end{pmatrix}$$

for all $a \in A$.

With suitable version of Voiculescu's Theorem [Elliott-Kucerovsky '01]:

Theorem

- $J_B \cong J_B \otimes \mathcal{K}$, and
- $\phi \oplus \pi \approx_u \phi$ (rel. J_B) and $\psi \oplus \pi \approx_u \psi$ (rel. J_B).

This allows us to remove π. Then $U \in B_\omega$ and $\phi \sim_u \psi$.

Summary

Theorem (S. '18)

Suppose

- A separable, unital, exact, UCT with faithful qd trace τ_A, and
- B simple, unital, $\mathcal{Q}$-stable AF-algebra with unique trace τ_B.

Embeddings $A \hookrightarrow B$ classified up to $\approx_u$ by K_0.

Remark (with Carrión, Gabe, Tikuisis, and White)

A version with much more general B also holds:
need B is simple, $\mathcal{Z}$-stable, unital, and exact (and more than K_0).

Corollary

Every separable, exact C*-algebra satisfying the UCT with a faithful quasidiagonal trace embeds into an AF-algebra.

Thank you!