Global asymptotics of particle systems at high temperature

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Based on preprint arXiv:2105.03795 — this is a joint work with Florent Benaych-Georges and Vadim Gorin.

Plan of the talk

Example 1: Hermite N–particle β–ensemble

Example 2: Spectra of sums of random matrices

Proof idea: Characteristic functions for correlated systems

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Hermite N–particle ensemble

The probability measure on $\mathbb{R}^N$ with density

$$\mathrm{Herm}(x_1, \cdots, x_N) \propto \prod_{1 \le i < j \le N} (x_i - x_j)^2 \prod_{k=1}^{N} e^{-x_k^2/2}$$

determines a random N-tuple of reals:

$$x_1 \le x_2 \le \cdots \le x_{N-1} \le x_N.$$

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The tuple $x_1 \le \cdots \le x_N$ is distributed like eigenvalues of the Hermitian random matrix

$$X = \frac{M + M^*}{2}, \quad M = [m_{ij}]_1^N, \quad m_{ij} = \mathcal{N}(0,1) + \sqrt{-1}\,\mathcal{N}(0,1).$$

The distribution of X is called Gaussian Unitary Ensemble (GUE) and $\mathrm{Herm}(\cdot)$ is the GUE (eigenvalue) density.

Global asymptotics of Hermite N–particle ensemble

To state the theorem, consider the empirical measures

$$\mu_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{\frac{x_i}{\sqrt{N}}}, \quad \text{where } x_1 \leq \cdots \leq x_N \text{ is GUE–distributed.}$$

Theorem (Wigner '55)

The (random) probability measures μ_N converge weakly, in probability, to the semicircle distribution — with density

$$\frac{\sqrt{4-t^2}}{2\pi}, \quad -2 \leq t \leq 2.$$

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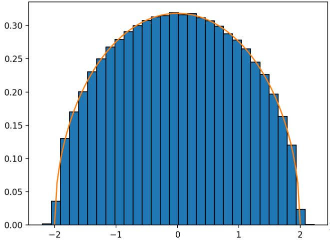
Graphically (figure by Yueqi Sheng):



Hermite N–particle β–ensemble

For general $\beta \geq 0$, the density

$$\mathrm{Herm}_{\boldsymbol{\beta}}(x_1, \cdots, x_N) \propto \prod_{1 \leq i < j \leq N} (x_i - x_j)^{\boldsymbol{\beta}} \prod_{k=1}^{N} e^{-x_k^2/2}$$

determines a random N-tuple of reals:

$$x_1 \leq x_2 \leq \cdots \leq x_{N-1} \leq x_N.$$

Reasons for the generalization:

1. For $\beta = 1$ ($\beta = 4$), this is the eigenvalue density of Gaussian Orthogonal (Symplectic) Ensemble: symmetric matrices with real entries (self-adjoint matrices with quaternionic entries).

2. In physics, distributions of this kind are known as log-gas systems and β is called the inverse temperature.

Global asymptotics of Hermite N–particle β–ensemble

Nothing changes if $\beta > 0$ is fixed: as $N \to \infty$, the empirical measures

$$\mu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{\frac{x_i}{\sqrt{N}}}, \quad \text{where } x_1 \leq \cdots \leq x_N \text{ is G}\beta\text{E–distributed},$$

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The outlier $\beta = 0$ case: In this case, the density is

$$\mathrm{Herm}_{\beta=0}(x_1, \cdots, x_N) \propto \prod_{k=1}^{N} e^{-x_k^2/2}.$$

Then $x_1, \cdots, x_N$ are i.i.d. standard Gaussian r.v.'s.
Hence if $\beta = 0$, and as $N \to \infty$, the measures $\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$ converge weakly, in probability, to the Gaussian distribution.

Global asymptotics of Hermite N–particle β–ensemble at high temperature

Theorem (Duy, Shirai '15 & Benaych-Georges, C, Gorin '21)

Consider the empirical measures

$$\mu_{N,\beta} := \frac{1}{N}\sum_{i=1}^{N}\delta_{\frac{x_i}{N}}, \quad \text{where } x_1 \le \cdots \le x_N \text{ is } G\beta E\text{–distributed.}$$

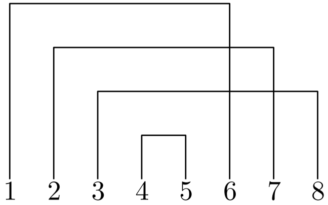
In the limit

$$N \to \infty, \quad \beta \to 0^+, \quad \frac{N\beta}{2} \to \gamma \in (0,\infty),$$

the measures $\mu_{N,\beta}$ converge weakly, in probability, to certain probability measure μ_γ which can be completely described.

Global asymptotics of Hermite N–particle β–ensemble at high temperature

For a *perfect matching* $\pi = \{B_1, \cdots, B_n\}$ of $\{1, \cdots, 2n\}$, draw the *arc diagram* and define $\mathrm{roof}(\pi)$ as the number of roofs with no intersections, e.g:



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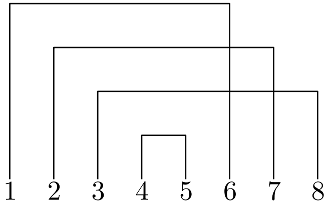

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Theorem (Benaych-Georges, Cuenca, Gorin '21)

The limiting measure μ_γ is uniquely determined by its moments:

$$\int_{-\infty}^{\infty} x^k \, \mu_\gamma(dx) = \sum_{\text{perfect matchings } \pi \text{ of } \{1,\cdots,k\}} (\gamma+1)^{\mathrm{roof}(\pi)}.$$

Limits as $\gamma \to 0^+$ and $\gamma \to \infty$

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Comments:

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2. If $k = 2n$ and $\gamma \to 0^+$, then

$$\begin{aligned} \text{RHS} &= \text{number of perfect matchings of } \{1, 2, \cdots, 2n\} \\ &= (2n-1)(2n-3)\cdots 3 \cdot 1. \end{aligned}$$

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3. If $k = 2n$ and $\gamma \to \infty$ (need to divide by γ^n first), then

RHS = number of *noncrossing* perfect matchings of $\{1, 2, \cdots, 2n\}$

$$= \text{Catalan number } \; C_n = \frac{(2n)!}{(n+1)!\,n!}.$$

Plan of the talk

Example 1: Hermite N–particle β–ensemble

Example 2: Spectra of sums of random matrices

Proof idea: Characteristic functions for correlated systems

Sums of complex Hermitian random matrices

Let $\mathbf{a} := (a_1 \leq \cdots \leq a_N)$ be an N–tuple of reals, and let A_N be a uniformly random complex Hermitian $N \times N$ matrix with spectra $\mathbf{a}$.

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Let $\mathbf{b} := (b_1 \leq \cdots \leq b_N)$ be an N–tuple of reals, and let B_N be a uniformly random complex Hermitian $N \times N$ matrix with spectra $\mathbf{b}$.

If A_N and B_N are independent: study the distribution of eigenvalues $\mathbf{c} := (c_1 \leq \cdots \leq c_N)$ of $C_N = A_N + B_N$.

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If A_N and B_N are independent: study the distribution of eigenvalues $\mathbf{c} := (c_1 \leq \cdots \leq c_N)$ of $C_N = A_N + B_N$.

We are interested in the global asymptotics, i.e. limits of $\frac{1}{N}\sum_{i=1}^{N} \delta_{c_i}$?

Sums of complex Hermitian random matrices

Theorem (Voiculescu '91)

Assume the weak limits:

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{a_i} \to \mu, \qquad \frac{1}{N}\sum_{i=1}^{N}\delta_{b_i} \to \nu.$$

Then we have the weak convergence, in probability:

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{c_i} \to \tau = \mu \boxplus \nu = \textit{free convolution of } \mu \textit{ and } \nu.$$

Moreover, τ can be computed efficiently.

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The computation of τ is given by:

- $R_\tau(z) = R_\mu(z) + R_\nu(z)$, where $R_\tau(z)$ is the R-transform of τ.
- If μ, ν are compactly supported, use moments, e.g.:

$$m_4^\tau = m_4^\mu + m_4^\nu + 4\, m_2^\mu m_2^\nu - 2\,(m_1^\mu)^2 (m_1^\nu)^2$$
$$+ 4\, m_1^\mu m_3^\nu + 4\, m_3^\mu m_1^\nu + 2\,(m_1^\mu)^2 m_2^\nu + 2\, m_2^\mu (m_1^\nu)^2.$$

β–analogue of sums of random matrices

The binary operation $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{c}$ has a natural β-analogue.

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Define the spherical (Harish-Chandra–Itzykson–Zuber) integral:

$$I_N(A, X) := \int_{U(N)} e^{\mathrm{Trace}(UAU^*X)} \mathrm{Haar}(dU) = \mathbb{E}\left[e^{\mathrm{Trace}(UAU^*X)} \right].$$

Here A, X are complex Hermitian, and integral is over the *Haar probability measure* on $U(N)$.

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Haar is $U(N)$-invariant $\Longrightarrow I_N(A, X)$ only depends on the (real) eigenvalues $\mathbf{a} = (a_1, \cdots, a_N)$, $\mathbf{x} = (x_1, \cdots, x_N)$ of A, X:

$$I_N(A, X) =: I_N(\mathbf{a}, \mathbf{x}).$$

β–analogue of sums of random matrices

Key fact: $\mathbf{c}$ is determined by:

$$\boxed{\mathbb{E}\left[\, I_N(\mathbf{c},\mathbf{x})\,\right] = I_N(\mathbf{a},\mathbf{x})\, I_N(\mathbf{b},\mathbf{x}), \text{ for all } \mathbf{x}.} \qquad (*)$$

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This is not hard to prove:

$$\begin{aligned}
I_N(\mathbf{a},\mathbf{x}) I_N(\mathbf{b},\mathbf{x}) &= \mathbb{E}\left[e^{\text{Trace}(U\text{diag}(\mathbf{a})U^* X)}\right] \cdot \mathbb{E}\left[e^{\text{Trace}(V\text{diag}(\mathbf{b})V^* X)}\right] \\
&= \mathbb{E}\left[e^{\text{Trace}(U\text{diag}(\mathbf{a})U^* X)} \cdot e^{\text{Trace}(V\text{diag}(\mathbf{b})V^* X)}\right] \\
&= \mathbb{E}\left[e^{\text{Trace}(C_N X)}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\text{Trace}(C_N X)} \,\middle|\, \text{Spec}(C_N) = \mathbf{c}\right]\right] \\
&= \mathbb{E}\left[\, I_N(\mathbf{c},\mathbf{x}) \,\right].
\end{aligned}$$

β–analogue of sums of random matrices

The function $I_N(\mathbf{a}, \mathbf{x})$ has a natural β-analogue $B_N^{\boldsymbol{\beta}}(\mathbf{a}, \mathbf{x})$, $\beta \geq 0$.

It is called the multivariate Bessel function.

- When $\beta = 2$: $B_N^{\boldsymbol{\beta}=2}(\mathbf{a}, \mathbf{x}) = I_N(\mathbf{a}, \mathbf{x})$.
- When $\beta = 1, 4$: $B_N^{\boldsymbol{\beta}=1}(\mathbf{a}, \mathbf{x}), B_N^{\boldsymbol{\beta}=4}(\mathbf{a}, \mathbf{x})$ are spherical integrals over compact groups (orthogonal and symplectic).

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- They are defined by eigenrelations for the (symmetrized) *Dunkl operators* P_k^{β}:

$$B_N^{\beta}(\mathbf{a}, \mathbf{x}) \text{ is symmetric in the variables } x_1, \cdots, x_N$$

$$P_k^{\beta} B_N^{\beta}(\mathbf{a}, \mathbf{x}) = \left(\sum_{i=1}^{N} a_i^k \right) \cdot B_N^{\beta}(\mathbf{a}, \mathbf{x}), \quad k = 1, 2, \cdots,$$

$$B_N^{\beta}(\mathbf{a}, \mathbf{0}) = 1.$$

A remark on the definition of β–sums of random matrices

Definition

The random N-tuple $\mathbf{c} := \mathbf{a} +_\beta \mathbf{b}$ *of reals is defined by:*

$$\boxed{\mathbb{E}\left[B_N^\beta(\mathbf{c},\mathbf{x})\right] = B_N^\beta(\mathbf{a},\mathbf{x})\, B_N^\beta(\mathbf{b},\mathbf{x}), \ \textit{for all } \mathbf{x}.} \qquad (*)$$

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Caveats:

- Existence of the random N-tuple $\mathbf{c} = \mathbf{a} +_\beta \mathbf{b}$ is a conjecture!
- $\mathbf{c}$ is a *distribution (generalized function)*. Our results are in this language, but to simplify notation in this talk we assume the conjecture holds: $\mathbf{c}$ is a random N-tuple.

Global asymptotics of β–sums of random matrices

Nothing changes if $\beta > 0$ is fixed: If $\mathbf{c} = \mathbf{a} +_\beta \mathbf{b}$ and the empirical measures of $\mathbf{a}$, $\mathbf{b}$ converge weakly to μ, ν. As $N \to \infty$, we have:

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{c_i} \to \tau = \mu \boxplus \nu = \text{free convolution of } \mu \text{ and } \nu.$$

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The outlier $\beta = 0$ case: In this case, the MBFs are

$$B_N^{\beta=0}(\mathbf{a}, \mathbf{x}) = \frac{1}{N!} \sum_{\sigma \in S(N)} e^{x_1 a_{\sigma(1)} + \cdots + x_N a_{\sigma(N)}}.$$

Then $\mathbf{c} = (c_1 \le \cdots \le c_N) := \mathbf{a} +_0 \mathbf{b}$ is the ordered tuple from $(a_1 + b_{\sigma(1)}, \cdots, a_N + b_{\sigma(N)})$, where $\sigma \in S(N)$ is uniformly random.

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As $N \to \infty$, we have:

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{c_i} \to \tau = \mu * \nu = \text{conventional convolution of } \mu \text{ and } \nu.$$

Global asymptotics of β–sums of random matrices at high temperature

Theorem (Benaych-Georges, Cuenca, Gorin '21)

Assume the weak limits:

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{a_i} \to \mu, \qquad \frac{1}{N}\sum_{i=1}^{N}\delta_{b_i} \to \nu,$$

where μ, ν *have compact support. Let* $\mathbf{c} := \mathbf{a} +_\beta \mathbf{b}$*. In the regime*

$$N \to \infty, \quad \beta \to 0^+, \quad \frac{N\beta}{2} \to \gamma,$$

we have the weak convergence, in probability:

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{c_i} \to \tau = \mu \boxplus_\gamma \nu =: \gamma\text{-convolution of } \mu \text{ and } \nu.$$

Describing the limiting measure

The moments m_k^τ are formulas in the moments m_i^μ, m_j^ν, e.g:

$$m_4^\tau = m_4^\mu + m_4^\nu + \left(4 + \frac{2}{\gamma+1}\right) m_2^\mu m_2^\nu - \frac{2\gamma}{\gamma+1}(m_1^\mu)^2(m_1^\nu)^2 + 4\, m_1^\mu m_3^\nu + 4\, m_3^\mu m_1^\nu + \frac{2\gamma}{\gamma+1}(m_1^\mu)^2\, m_2^\nu + \frac{2\gamma}{\gamma+1} m_2^\mu\, (m_1^\nu)^2,$$

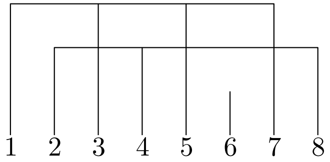
but the expressions are complicated!

Describing the limiting measure

The heroes are the γ-cumulants $\kappa_1, \kappa_2, \cdots$ defined as follows:

- For a *set partition* $\pi = \{B_1, \cdots, B_m\}$ of $\{1, 2, \cdots, k\}$, draw the *arc diagram* of π and define the weight

$$W_\gamma(\pi) := \prod_{i=1}^{m} p(i)!\,(\gamma + p(i) + 1)_{|B_i|-1-p(i)}.$$



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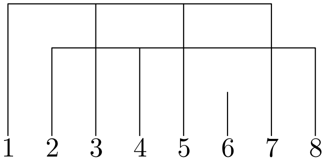

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- Then $\kappa_1, \kappa_2, \cdots$ are *recursively* defined by

$$m_k = \sum_{\text{set partitions } \pi \text{ of } \{1,\cdots,k\}} W_\gamma(\pi) \prod_{B\in\pi} \kappa_{|B|}, \quad k = 1, 2, \cdots.$$

Describing the limiting measure

γ-cumulants and moments encode the same information:

$$m_1 = \kappa_1$$

$$m_2 = (\gamma + 1)\kappa_2 + \kappa_1^2$$

$$m_3 = (\gamma + 1)(\gamma + 2)\kappa_3 + 3(\gamma + 1)\kappa_2\kappa_1 + \kappa_1^3$$

$$m_4 = (\gamma + 1)(\gamma + 2)(\gamma + 3)\kappa_4 + (\gamma + 1)(2\gamma + 3)\kappa_2^2 + \cdots$$

$$\cdots$$

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$$\cdots$$

Theorem (Benaych-Georges, Cuenca, Gorin '21)

The limiting measure τ is compactly supported, and it's determined by its γ-cumulants $\kappa_1^\tau, \kappa_2^\tau, \cdots$:

$$\kappa_n^\tau = \kappa_n^\mu + \kappa_n^\nu, \quad n = 1, 2, \cdots .$$

Plan of the talk

Example 1: Hermite N–particle β–ensemble

Example 2: Spectra of sums of random matrices

Proof idea: Characteristic functions for correlated systems

Recall: Levy's continuity theorem

Assume that $\{\mu_N\}_{N\geq 1}$, and μ are probability measures on $\mathbb{R}^d$.

The *characteristic function = Fourier transform* of μ_N is defined as

$$\phi_N(x_1, \cdots, x_d) := \int_{\mathbb{R}^d} e^{\mathbf{i}(t_1 x_1 + \cdots + t_d x_d)} \mu_N(t_1, \cdots, t_d).$$

Similarly, $\phi(x_1, \cdots, x_d)$ is the Fourier transform of μ.

Theorem

$\mu_N \to \mu$ *weakly* $\Leftrightarrow \phi_N(x_1, \cdots, x_d) \to \phi(x_1, \cdots, x_d)$ *pointwise.*

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Assume that $\{\mu_N\}_{N\geq 1}$, and μ are probability measures on $\mathbb{R}^d$.

The *characteristic function = Fourier transform* of μ_N is defined as

$$\phi_N(x_1, \cdots, x_d) := \int_{\mathbb{R}^d} e^{\mathbf{i}(t_1 x_1 + \cdots + t_d x_d)} \mu_N(t_1, \cdots, t_d).$$

Similarly, $\phi(x_1, \cdots, x_d)$ is the Fourier transform of μ.

Theorem

$\mu_N \to \mu$ *weakly* $\Leftrightarrow \phi_N(x_1, \cdots, x_d) \to \phi(x_1, \cdots, x_d)$ *pointwise.*

Application: When μ_N is the distribution of the average $\frac{X_1 + \cdots + X_N}{N}$ of *independent* identically distributed r.v's.

Philosophy: Convergence of measures is equivalent to convergence of functions (Fourier transforms).

Bessel generating function = symmetric Dunkl transform

For correlated systems of the random matrix type, the (symmetric) Dunkl transform is more natural:

$$
\begin{aligned}
G_N^{\beta}(x_1, \cdots, x_N) &:= \int_{\mathbb{R}^N} B_N^{\beta}\left((x_1, \cdots, x_N), (t_1, \cdots, t_N)\right) \mu_N(t_1, \cdots, t_N) \\
&= \mathbb{E}_{\mu_N}\left[B_N^{\beta}\left((x_1, \cdots, x_N), (t_1, \cdots, t_N)\right)\right].
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$$= \mathbb{E}_{\mu_N}\left[B_N^\beta\left((x_1,\cdots,x_N),(t_1,\cdots,t_N)\right)\right].$$

The idea is:

$$B_N^\beta\left((x_1,\cdots,x_N),(t_1,\cdots,t_N)\right) \Longleftarrow e^{\mathbf{i}(t_1x_1+\cdots+t_Nx_N)}$$
$$P_k^\beta \Longleftarrow \frac{\partial^k}{\partial x_1^k}+\cdots+\frac{\partial^k}{\partial x_N^k}$$

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Implies:

$$\boxed{P_k^\beta\left(G_N^\beta\right)\Big|_{x_1=\cdots=x_N=0} = \mathbb{E}_{\mu_N}\left[t_1^k+\cdots+t_N^k\right]} \quad (**)$$

Our general approach

$$\boxed{P_k^{\beta}\left(e^{\ln(G_N^{\beta})}\right)\Big|_{x_1=\cdots=x_N=0} = \mathbb{E}_{\mu_N}\left[t_1^k+\cdots+t_N^k\right]} \quad (**)$$

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1. Equations like (**) link probabilistic information of μ_N, e.g.

$$\lim_{N\to\infty}\mathbb{E}_{\mu_N}\left[\frac{t_1^k+\cdots+t_N^k}{N}\right] =: m_k, \quad k=1,2,\cdots,$$

and analytic information of G_N^β, e.g.

$$\lim_{\substack{N\to\infty,\,\beta\to 0\\ N\beta\to 2\gamma}} \frac{1}{(s-1)!}\frac{\partial^s}{\partial x_1^s}\left(\ln(G_N^\beta)\right)\Big|_{x_1=\cdots=x_N=0} =: \kappa_s, \quad s=1,2,\cdots.$$

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2. The relation between sequences $m_1, m_2, \cdots$ and $\kappa_1, \kappa_2, \cdots$ is given by generating functions and operators — however, this is transformed into the combinatorial relations:

$$m_k = \sum_{\text{set partitions } \pi \text{ of } \{1,\cdots,k\}} W_{\gamma}(\pi)\prod_{B\in\pi}\kappa_{|B|}, \quad k=1,2,\cdots.$$

Conclusion

- In the spirit of Levy's continuity theorem:

$$\underset{\text{if } (x_1,\cdots,x_N) \text{ is } \mu_N\text{-distributed}}{\tfrac{1}{N}\sum_{i=1}^{N}\delta_{x_i}\to\{m_1,m_2,\cdots\}} \Leftrightarrow G_N^{\beta}(x_1,x_2,\cdots,x_N) \to \{\kappa_1,\kappa_2,\cdots\}.$$

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- For μ_N being the distribution of $\mathbf{c}=\mathbf{a}+_\beta\mathbf{b}$:

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Thank you for your attention!