Strong convergence of tensor products of independent G.U.E. matrices

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Based on arXiv:2205.07695v1 (preliminary version)

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UC Berkeley Probabilistic Operator Algebra Seminar
October 3rd 2022

Theory of Operator Algebras
Free Probability Theory

$\updownarrow$

Random Matrix Theory

Theory of Operator Algebras
Free Probability Theory

$\uparrow$ this work (via a previous work of B. Hayes)

Random Matrix Theory

G.U.E. (Gaussian Unitary Ensemble) matrix W_N

Definition

A $N \times N$ G.U.E. matrix is a selfadjoint random matrix $W_N = (W_N)_{1 \leq i,j \leq N}$ such that

$(W_N)_{i,i}, 1 \leq i \leq N$, and $\sqrt{2}\Re(W_N)_{i,j}, \sqrt{2}\Im(W_N)_{i,j}$, $1 \leq j < i \leq N$ are independent standard Gaussian random variables.

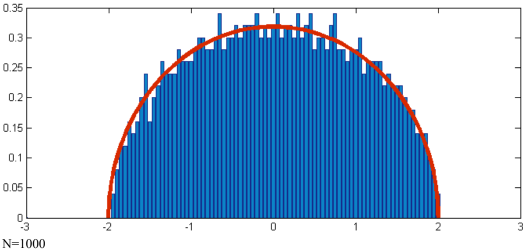
We call $X = \frac{W_N}{\sqrt{N}}$ a *normalized* G.U.E.

$\lambda_i(\frac{W_N}{\sqrt{N}})$, $i = 1, \ldots, N$: the eigenvalues of $\frac{W_N}{\sqrt{N}}$.

Theorem (Wigner (50'))

$$\mu_{\frac{W_N}{\sqrt{N}}} := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(\frac{W_N}{\sqrt{N}})} \to \mu_{sc} \quad a.s \text{ when } \quad N \to +\infty$$

$$\textit{semicircular law } \frac{d\mu_{sc}}{dx}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\, 1_{[-2,2]}(x)$$



Semicircular noncommutative random variable

Definition

$(\mathcal{A}, \phi)$ a C^*-noncommutative probability space. A standard semicircular noncommutative random variable s in $(\mathcal{A}, \phi)$ is a selfadjoint operator in $\mathcal{A}$ such that

$$\forall k \in \mathbb{N},\ \phi(s^k) = \int x^k d\mu_{sc}(x),$$

$$\frac{d\mu_{sc}}{dx}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\, 1_{[-2,2]}(x).$$

$$\forall k,\ \frac{1}{N}\mathrm{Tr}_N\left[\left(\frac{W_N}{\sqrt{N}}\right)^k\right] \to \phi(s^k) \text{ a.s when } N \to +\infty.$$

Theorem ("Asymptotic freeness", Voiculescu (91), Thorbjørnsen (00))

Let $X_j, 1 \leq j \leq r$ be independent normalized GUE $N \times N$ matrices. Let $(s_1, \ldots, s_r)$ be a standard free semicircular system in $(\mathcal{A}, \phi)$. Then, almost surely for any noncommutative polynomial P in r variables,

$$\lim_{N \to +\infty} \frac{1}{N} Tr_N \left(P(X_1, \ldots, X_r) \right) = \phi \left(P(s_1, \ldots, s_r) \right).$$

Definition

A *standard free semicircular system* in $(\mathcal{A}, \phi)$ is any tuple of r free standard semicircular operators $(s_1, \ldots, s_r)$.

Theorem ("Asymptotic freeness", Voiculescu (91), Thorbjørnsen (00))

Let $X_j, 1 \leq j \leq r$ be independent normalized GUE $N \times N$ matrices. Let $(s_1, \ldots, s_r)$ be a standard free semicircular system in $(\mathcal{A}, \phi)$. Then, almost surely for any noncommutative polynomial P in r variables,

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Theorem ("Strong asymptotic freeness", Haagerup-Thorbjørnsen (05))

Almost surely for any noncommutative polynomial P in r variables,

$$\lim_{N \to +\infty} \|P(X_1, \ldots, X_r)\| = \|P(s_1, \ldots, s_r)\|.$$

$\Longrightarrow$ numerous consequences for operator algebras issues.

An operator algebra conjecture

Conjecture (Peterson-Thom '11)

Let $\mathbb{F}_r$ be the free group with r free generators, and $L(\mathbb{F}_r)$ its group von Neumann algebra. Assume that $Q \subset L(\mathbb{F}_r)$ is a diffuse, amenable von Neumann subalgebra. Then there exists a unique maximal amenable von Neumann subalgebra $P \subset L(\mathbb{F}_r)$ such that $Q \subseteq P$.

Motivation: a result of Ben Hayes

Theorem (Ben Hayes 2020)

$X_N^{(1)}, \ldots, X_N^{(r)}, Y_N^{(1)}, \ldots, Y_N^{(r)}$: $N \times N$ independent normalized G.U.E., I_N: the $N \times N$ identity matrix. $(\mathbf{s}_1, \ldots, \mathbf{s}_r)$: a free standard semicircular system, $1_{\mathbf{s}}$: the unit of $C^*(\mathbf{s}_1, \ldots, \mathbf{s}_r)$, the unital C^*-algebra generated by the free semicirculars $\mathbf{s}_1, \ldots, \mathbf{s}_r$. If almost surely, for any polynomial P in $2r$ selfadjoint noncommuting indeterminates,

$$\lim_{N\to\infty} \left\| P(X_N^{(1)} \otimes I_N, \ldots, X_N^{(r)} \otimes I_N, I_N \otimes Y_N^{(1)}, \ldots, I_N \otimes Y_N^{(r)}) \right\| = \| P(\mathbf{s}_1 \otimes 1_{\mathbf{s}}, \ldots, \mathbf{s}_r \otimes 1_{\mathbf{s}}, 1_{\mathbf{s}} \otimes \mathbf{s}_1, \ldots, 1_{\mathbf{s}} \otimes \mathbf{s}_r) \|_{\min},$$

then Peterson-Thom's conjecture is true.

Some previous works

Recent works established strong convergence of matrices

$$X_N^{(1)} \otimes I_M, \ldots, X_N^{(r)} \otimes I_M, I_N \otimes Y_M^{(1)}, \ldots, I_N \otimes Y_M^{(r)}$$

where the dimension of the G.U.E. matrices $Y_M^{(i)}$'s is M

- $M = o(N^{1/4})$ (Pisier 2014),
- $M = o(N^{1/3})$ (Collins-Guionnet-Parraud 2019),
- $M = o(N/(\log N)^3)$ (Bandeira-Boedihardjo-Van Handel 2021).

This does not suffice for the purpose of Ben Hayes, which requires $M = N$.

Theorem (B.-C. (2022))

$X_N^{(1)}, \ldots, X_N^{(r_1)}, Y_N^{(1)}, \ldots, Y_N^{(r_2)}$:$N \times N$ *independent normalized G.U.E.,*

$(\mathbf{s}_1, \ldots, \mathbf{s}_{r_1})$, $(\mathbf{t}_1, \ldots, \mathbf{t}_{r_2})$: *two standard semicircular systems in a C^*-probability space endowed with a faithful tracial state $(\mathcal{A}, \phi)$,*

Almost surely, for any polynomial P in $r_1 + r_2$ selfadjoint noncommuting indeterminates,

$$\lim_{N\to\infty} \left\| P(X_N^{(1)} \otimes I_N, \ldots, X_N^{(r_1)} \otimes I_N, I_N \otimes Y_N^{(1)}, \ldots, I_N \otimes Y_N^{(r_2)}) \right\| \\ = \| P(\mathbf{s}_1 \otimes 1_{\mathcal{A}}, \ldots, \mathbf{s}_{r_1} \otimes 1_{\mathcal{A}}, 1_{\mathcal{A}} \otimes \mathbf{t}_1, \ldots, 1_{\mathcal{A}} \otimes \mathbf{t}_{r_2}) \|_{\min} .$$

Note that, obviously, the asymptotic freeness of the $X_N^{(i)}$'s and the asymptotic freeness of the $Y_N^{(i)}$'s imply that, almost surely, for any polynomial P in $r_1 + r_2$ selfadjoint noncommuting indeterminates,

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N^2} \mathrm{Tr}_N \otimes \mathrm{Tr}_N \left(P(X_N^{(1)} \otimes I_N, \ldots, X_N^{(r_1)} \otimes I_N, I_N \otimes Y_N^{(1)}, \ldots, I_N \otimes Y_N^{(r_2)}) \right) \\ = \tau \otimes \tau \left(P(\mathbf{s}_1 \otimes 1_{\mathcal{A}}, \ldots, \mathbf{s}_{r_1} \otimes 1_{\mathcal{A}}, 1_{\mathcal{A}} \otimes \mathbf{t}_1, \ldots, 1_{\mathcal{A}} \otimes \mathbf{t}_{r_2}) \right),\end{aligned}$$

by using that $(X_N^{(i)} \otimes I_N)$ commutes with $I_N \otimes Y_N^{(j)}$ and $\mathbf{s}_i \otimes 1_{\mathcal{A}}$ commutes with $1_{\mathcal{A}} \otimes \mathbf{t}_j$.

Skeleton of the proof of the convergence of norms.

First step: from GUE to unitary matrices

Theorem (Yin (2017))

$(\mathcal{A}_n, \tau_n)$, $(\mathcal{A}, \tau)$ faithful tracial C^*-probability spaces. Assume that $x(n) = (x_1(n), \ldots, x_r(n)) \in (\mathcal{A}_n, \tau_n)$ strongly converges to $x = (x_1, \ldots, x_r)$ in $(\mathcal{A}, \tau)$ and that the tuple (x, x^*) lies in the domain of a rational function R. Then we have

- *$R(x(n), (x(n))^*)$ is well defined for sufficiently large n;*
- *the convergence of traces and norms,*

$$\lim_{n \to +\infty} \tau_n(R(x(n), x(n)^*)) = \tau(R(x, x^*)),$$

$$\lim_{n \to +\infty} \|R(x(n), x(n)^*)\|_{\mathcal{A}_n} = \|R(x, x^*)\|_{\mathcal{A}}.$$

Recall that $x(n)$ strongly converges to $x = (x_1, \ldots, x_r)$ means that for any polynomial P in $2r$ non-commuting indeterminates,

$$\lim_{n \to +\infty} \tau_n(P(x(n), x(n)^*)) = \tau(P(x, x^*)),$$
$$\lim_{n \to +\infty} \|P(x(n), x(n)^*)\|_{\mathcal{A}_n} = \|P(x, x^*)\|_{\mathcal{A}}.$$

Idea: use the Cayley transform, $\Psi : \mathbb{R} \to \mathbb{T} \setminus \{1\}, \Psi(x) = \frac{x+i}{x-i}, \Psi^{-1}(w) = i\frac{w+1}{w-1}$

$$(\Psi(X_N^{(1)}), \dots, \Psi(X_N^{(r_1)})) =: (U_1, \dots, U_{r_1}) =: U,$$

$$(\Psi(\mathbf{s}_1), \dots, \Psi(\mathbf{s}_{r_1})) =: (u_1, \dots, u_{r_1}) =: u,$$

$$(\Psi(Y_N^{(1)}), \dots, \Psi(Y_N^{(r_2)})) =: (V_1, \dots, V_{r_2}) =: V,$$

$$(\Psi(\mathbf{t}_1), \dots, \Psi(\mathbf{t}_{r_2})) =: (v_1, \dots, v_{r_2}) =: v,$$

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$$(\Psi(\mathbf{t}_1), \dots, \Psi(\mathbf{t}_{r_2})) =: (v_1, \dots, v_{r_2}) =: v,$$

Theorem (B.-C.(2022))

Almost surely, for any polynomial P in $r_1 + r_2$ variables and their adjoints, $\|P(U \otimes I_N, I_N \otimes V, U^* \otimes I_N, I_N \otimes V^*)\|$ converges to $\|P(u \otimes 1_{\mathcal{A}}, 1_{\mathcal{A}} \otimes v, u^* \otimes 1_{\mathcal{A}}, 1_{\mathcal{A}} \otimes v^*)\|$.

Notation for short: $U \otimes I = (U_1 \otimes I, \dots, U_{r_1} \otimes I)$ and $I \otimes V = (I \otimes V_1, \dots, I \otimes V_{r_1})$

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Notice that any noncommutative polynomial in $(X \otimes I_N, I_N \otimes Y)$ is a rational function in $(U \otimes I_N, I_N \otimes V)$

Yin's result
$\downarrow$
$\Longrightarrow$ strong convergence of tensor products of G.U.E. matrices.

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Theorem (B.-C.(2022))

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Yin's result
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$\Longrightarrow$ strong convergence of tensor products of G.U.E. matrices.

Advantage: The U_i's and V_i's are unitary matrices whereas the sequence of G.U.E. matrices is not bounded.

Almost surely, for any polynomial P in $r_1 + r_2$ noncommuting indeterminates and their adjoints,

$$\lim_{N\to\infty} \frac{1}{N^2} \mathrm{Tr}_N \otimes \mathrm{Tr}_N \left(P(U \otimes I_N, I_N \otimes V, U^* \otimes I_N, I_N \otimes V^*)\right)$$
$$= \tau \otimes \tau \left(P(u \otimes 1_{\mathcal{A}}, 1_{\mathcal{A}} \otimes v, u^* \otimes 1_{\mathcal{A}}, 1_{\mathcal{A}} \otimes v^*)\right),$$

$\Rightarrow$ almost surely for any P,

$$\liminf_{N\to+\infty} \|P(U \otimes I_N, I_N \otimes V, U^* \otimes I_N, I_N \otimes V^*)\|$$
$$\geq \left\|P\left(u \otimes 1_{\mathcal{A}}, 1_{\mathcal{A}} \otimes v, u^* \otimes 1_{\mathcal{A}}, 1_{\mathcal{A}} \otimes v^*\right)\right\|.$$

Thus, easy: almost surely for any P,

$$\liminf_{N\to+\infty} \|P(U\otimes I_N, I_N\otimes V, U^*\otimes I_N, I_N\otimes V^*)\|$$

$$\geq \|P(u\otimes 1_{\mathcal{A}}, 1_{\mathcal{A}}\otimes v, u^*\otimes 1_{\mathcal{A}}, 1_{\mathcal{A}}\otimes v^*)\|.$$

The difficulty: proving that almost surely for any P,

$$\limsup_{N\to+\infty} \|P(U\otimes I_N, I_N\otimes V, U^*\otimes I_N, I_N\otimes V^*)\|$$

$$\leq \|P(u\otimes 1_{\mathcal{A}}, 1_{\mathcal{A}}\otimes v, u^*\otimes 1_{\mathcal{A}}, 1_{\mathcal{A}}\otimes v^*)\|.$$

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$$\leq \|P(u\otimes 1_{\mathcal{A}}, 1_{\mathcal{A}}\otimes v, u^*\otimes 1_{\mathcal{A}}, 1_{\mathcal{A}}\otimes v^*)\|.$$

Our proof uses the **linearization trick** developped in 2005 by Haagerup-Thorbjornsen and Schultz when they proved strong asymptotic freeness of independent GUE/GOE matrices:

in order to prove the above inequality involving any polynomial, it suffices to prove an inclusion of spectra of any selfadjoint **affine** polynomial but **with coefficients in** $M_m(\mathbb{C})$, for any $m \in \mathbb{N}$.

2nd step: Linearization trick, Haagerup-Thorbjornsen and Schultz's method

$U_i \otimes I_N$ $U_i^* \otimes I_N$

$I_N \otimes V_j$ $I_N \otimes V_j^*$

2nd step: Linearization trick, Haagerup-Thorbjornsen and Schultz's method

For all $m \in \mathbb{N}$, all matrices $\xi = \xi^*, \gamma_1, \ldots, \gamma_{r_1}, \beta_1, \ldots, \beta_{r_2} \in M_m(\mathbb{C})$

$$
\begin{aligned}
S_N \quad = \quad & \xi \otimes I_N \otimes I_N + \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes I_N + \sum_{i=1}^{r_1} \gamma_i^* \otimes U_i^* \otimes I_N \\
& + \sum_{j=1}^{r_2} \beta_j \otimes I_N \otimes V_j + \sum_{j=1}^{r_2} \beta_j^* \otimes I_N \otimes V_j^*
\end{aligned}
$$

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$$\begin{aligned} S_N &= \xi \otimes I_N \otimes I_N + \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes I_N + \sum_{i=1}^{r_1} \gamma_i^* \otimes U_i^* \otimes I_N \\ &\quad + \sum_{j=1}^{r_2} \beta_j \otimes I_N \otimes V_j + \sum_{j=1}^{r_2} \beta_j^* \otimes I_N \otimes V_j^* \\ \mathcal{S} &= \xi \otimes 1_{\mathcal{A}} \otimes 1_{\mathcal{A}} + \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_{\mathcal{A}} + \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_{\mathcal{A}} \\ &\quad + \sum_{j=1}^{r_2} \beta_j \otimes 1_{\mathcal{A}} \otimes v_j + \sum_{j=1}^{r_2} \beta_j^* \otimes 1_{\mathcal{A}} \otimes v_j^* \end{aligned}$$

2nd step: Linearization trick, Haagerup-Thorbjornsen and Schultz's method

Proposition (B.-C. (2022))

For all $m \in \mathbb{N}$, all matrices $\xi = \xi^*, \gamma_1, \dots, \gamma_{r_1}, \beta_1, \dots, \beta_{r_2} \in M_m(\mathbb{C})$ and all $\varepsilon > 0$, almost surely, for all large N,

$$\mathrm{sp}\,(S_N) \subset \mathrm{sp}\,(\mathcal{S}) + (-\varepsilon, \varepsilon),$$

$$\begin{aligned} S_N &= \xi \otimes I_N \otimes I_N + \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes I_N + \sum_{i=1}^{r_1} \gamma_i^* \otimes U_i^* \otimes I_N \\ &\quad + \sum_{j=1}^{r_2} \beta_j \otimes I_N \otimes V_j + \sum_{j=1}^{r_2} \beta_j^* \otimes I_N \otimes V_j^* \\ \mathcal{S} &= \xi \otimes 1_{\mathcal{A}} \otimes 1_{\mathcal{A}} + \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_{\mathcal{A}} + \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_{\mathcal{A}} \\ &\quad + \sum_{j=1}^{r_2} \beta_j \otimes 1_{\mathcal{A}} \otimes v_j + \sum_{j=1}^{r_2} \beta_j^* \otimes 1_{\mathcal{A}} \otimes v_j^* \end{aligned}$$

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$$\mathrm{sp}\,(S_N) \subset \mathrm{sp}\,(\mathcal{S}) + (-\varepsilon, \varepsilon),$$

$$\begin{aligned} S_N &= \xi \otimes I_N \otimes I_N + 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes I_N + 2\Re \sum_{j=1}^{r_2} \beta_j \otimes I_N \otimes V_j \\ \mathcal{S} &= \xi \otimes 1_{\mathcal{A}} \otimes 1_{\mathcal{A}} + 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_{\mathcal{A}} + 2\Re \sum_{j=1}^{r_2} \beta_j \otimes 1_{\mathcal{A}} \otimes v_j \end{aligned}$$

This inclusion of spectra of affine polynomials with matrix-valued coefficients implies, by Haagerup-Thorbjornsen's arguments, the desired reverse inequalities: almost surely for any P,

$$\limsup_{N\to+\infty} \|P(U \otimes I_N, I_N \otimes V, U^* \otimes I_N, I_N \otimes V^*)\|$$

$$\leq \|P\left(u \otimes 1_{\mathcal{A}}, 1_{\mathcal{A}} \otimes v, u^* \otimes 1_{\mathcal{A}}, 1_{\mathcal{A}} \otimes v^*\right)\|.$$

Proof of the inclusion of spectra

$$\forall z \in \mathbb{C} \setminus \mathbb{R}, \quad g_N(z) = \mathbb{E}(\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \mathrm{tr}_N)[(zI_m \otimes I_N \otimes I_N - S_N)^{-1}],$$
$$g(z) = (\mathrm{tr}_m \otimes \tau \otimes \tau)[(zI_m \otimes 1_{\mathcal{A}} \otimes 1_{\mathcal{A}} - \mathcal{S})^{-1}].$$

Proposition (B.-C. (2022))

There exists a polynomial Q with nonnegative coefficients such that, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\left| g_N(z) - g(z) - \frac{E(z)}{N^2} \right| \leq \frac{Q(|\Im z|^{-1})}{N^4},$$

where $E(z) = \Lambda\left(\frac{1}{z-x}\right)$, Λ is a distribution whose support is included in the spectrum of $\mathcal{S}$ and $\Lambda(1) = 0$.

(By a method developped by H-T-S) $\Longrightarrow$ almost surely, for all large N,

$$\mathrm{sp}\,(S_N) \subset \mathrm{sp}\,(\mathcal{S}) + (-\varepsilon, \varepsilon).$$

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$\Updownarrow$ Thanks to Yin's result

Strong convergence of tensorized Cayley transforms U_i's, V_j's of G.U.E.'s

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Strong convergence of tensorized Cayley transforms U_i's, V_j's of G.U.E.'s

$\Uparrow$ Thanks to H-T-S's approach

$$\forall z \in \mathbb{C} \setminus \mathbb{R}, \quad \left| g_N(z) - g(z) - \frac{E(z)}{N^2} \right| \leq \frac{Q(|\Im z|^{-1})}{N^4},$$

E: Stieltjes transform of Λ, support(Λ) $\subset$ spect($\mathcal{S}$),$\Lambda(1) = 0$,

$$g_N(z) = \mathbb{E}(\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \mathrm{tr}_N)[(zI_m \otimes I_N \otimes I_N - S_N)^{-1}],$$

$$g(z) = (\mathrm{tr}_m \otimes \tau \otimes \tau)[(zI_m \otimes 1_{\mathcal{A}} \otimes 1_{\mathcal{A}} - \mathcal{S})^{-1}],$$

$$S_N = \xi \otimes I_N \otimes I_N + \sum_{i=1}^{r_1} 2\Re(\gamma_i \otimes U_i \otimes I_N) + \sum_{j=1}^{r_2} 2\Re(\beta_j \otimes I_N \otimes V_j),$$

$$\mathcal{S} = \xi \otimes 1_{\mathcal{A}} \otimes 1_{\mathcal{A}} + \sum_{i=1}^{r_1} 2\Re(\gamma_i \otimes u_i \otimes 1_{\mathcal{A}}) + \sum_{j=1}^{r_2} 2\Re(\beta_j \otimes 1_{\mathcal{A}} \otimes v_j).$$

Strategy

Establish for any $z \in \mathbb{C} \setminus \mathbb{R}$, $|z|$ large enough, for any N, that

$$g_N(z) = g(z) + \frac{E(z)}{N^2} + \frac{\Delta_N(z)}{N^4} \qquad (1)$$

with explicit formulae for E and Δ_N, E and Δ_N having analytic extension on $\mathbb{C} \setminus \mathbb{R}$.

$\Longrightarrow$ Extend the identity (1) by analyticity to all $z \in \mathbb{C} \setminus \mathbb{R}$.

For $|z|$ large, a series expansion to decouple U and V

$|z| > \|\xi\| + 2\sum_{i=1}^{r_1}\|\gamma_i\| + 2\sum_{i=1}^{r_2}\|\beta_i\|$, then

$$\left((zI_m - \xi) \otimes I_N \otimes I_N - 2\Re\sum_{i=1}^{r_1}\gamma_i \otimes U_i \otimes I_N - 2\Re\sum_{i=1}^{r_2}\beta_i \otimes I_N \otimes V_i\right)^{-1}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\xi \otimes I_N \otimes I_N + 2\Re\sum_{i=1}^{r_1}\gamma_i \otimes U_i \otimes I_N + 2\Re\sum_{i=1}^{r_2}\beta_i \otimes I_N \otimes V_i\right)^n}{z^{n+1}}$$

For $|z|$ large, a series expansion to decouple U and V

$|z| > \|\xi\| + 2\sum_{i=1}^{r_1}\|\gamma_i\| + 2\sum_{i=1}^{r_2}\|\beta_i\|$, then

$$\left((zI_m - \xi)\otimes I_N \otimes I_N - 2\Re\sum_{i=1}^{r_1}\gamma_i\otimes U_i\otimes I_N - 2\Re\sum_{i=1}^{r_2}\beta_i\otimes I_N\otimes V_i\right)^{-1}$$

$$= \sum_{n=0}^{\infty}\frac{\left(\xi\otimes I_N\otimes I_N + 2\Re\sum_{i=1}^{r_1}\gamma_i\otimes U_i\otimes I_N + 2\Re\sum_{i=1}^{r_2}\beta_i\otimes I_N\otimes V_i\right)^n}{z^{n+1}}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{\substack{0\le i_1,\dots,i_n\le r_2 \\ k \text{ of the } i\text{'s being equal to } 0, \\ \text{and the other } n-k \text{ being } i_{\iota_1},\dots,i_{\iota_{n-k}}}}\ \sum_{\epsilon_{\iota_1},\dots,\epsilon_{\iota_{n-k}}\in\{\pm 1\}}\frac{\tilde{T}_{i_1}\cdots\tilde{T}_{i_n}\otimes\left(V_{i_{\iota_1}}^{\epsilon_{\iota_1}}\cdots V_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}}\right)}{z^{n+1}}.$$

$$\tilde{T}_{i_j}\in\{\beta_{i_j}\otimes I_N, \beta_{i_j}^*\otimes I_N\} \text{ if } i_j\neq 0,\ \tilde{T}_0 = \xi\otimes I_N + 2\Re\sum_{i=1}^{r_1}\gamma_i\otimes U_i.$$

For $|z|$ large, a series expansion to decouple U and V

$|z| > \|\xi\| + 2\sum_{i=1}^{r_1} \|\gamma_i\| + 2\sum_{i=1}^{r_2} \|\beta_i\|$, then

$$g_N(z) =$$

$$\mathbb{E}\,\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \mathrm{tr}_N \left((zI_m - \xi) \otimes I_N \otimes I_N - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes I_N - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes V_i \right)^{-1}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{\substack{0 \leq i_1, \ldots, i_n \leq r_2 \\ k \text{ of the } i\text{'s being equal to } 0, \\ \text{and the other } n-k \text{ being } i_{\iota_1}, \ldots, i_{\iota_{n-k}}}} \sum_{\epsilon_{\iota_1}, \ldots, \epsilon_{\iota_{n-k}} \in \{\pm 1\}} \frac{\mathbb{E}\,\mathrm{tr}_m \otimes \mathrm{tr}_N \left(\tilde{T}_{i_1} \cdots \tilde{T}_{i_n} \right)}{z^{n+1}}$$

$$\times\, \mathbb{E}\,\mathrm{tr}_N \left(V_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots V_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}} \right).$$

$\Longrightarrow$ Idea: use a precise expansion in $1/N$ of the mixed moments of the V_i's and their adjoints

Extension of Parraud's formula to polynomial in Cayley transforms V_i's of GUE's

Proposition (B.-C. 2022)

$$\begin{aligned}
&\mathbb{E}\mathrm{tr}_N(P(V_1, \ldots, V_r, V_1^*, \ldots, V_r^*)) \\
&= \tau(P(v_1, \ldots, v_r, v_1^*, \ldots, v_r^*)) + \frac{\nu_1^{(N)}(P)}{N^2} \\
&= \tau(P(v_1, \ldots, v_r, v_1^*, \ldots, v_r^*)) + \frac{\nu_1(P)}{N^2} + \frac{\nu_2^{(N)}(P)}{N^4}
\end{aligned}$$

where $\nu_1^{(N)}$, ν_1, $\nu_2^{(N)}$ *are explicit. (The* v_i*'s are Cayley transforms of the semi-circular variables* t_i*'s).*

Fundamental tool: Félix Parraud's formulae for independent GUE matrices $X_1, \ldots, X_r$ (2020).

For $|z|$ large, a series expansion to decouple U and V

$|z| > \|\xi\| + 2\sum_{i=1}^{r_1} \|\gamma_i\| + 2\sum_{i=1}^{r_2} \|\beta_i\|$, then

$$\mathbb{E}\,\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \mathrm{tr}_N \left((zI_m - \xi)\otimes I_N \otimes I_N - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes I_N - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes V_i\right)^{-1}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \sum_{0 \leq i_1, \dots, i_n \leq r_2} \sum_{\epsilon_{\iota_1}, \dots, \epsilon_{\iota_{n-k}} \in \{\pm 1\}} \frac{\left\{\mathbb{E}\,\mathrm{tr}_m \otimes \mathrm{tr}_N (\tilde{T}_{i_1} \cdots \tilde{T}_{i_n})\right\} \mathbb{E}\,\mathrm{tr}_N (V_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots V_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}})}{z^{n+1}}$$

k of the i's being equal to 0,

and the other $n-k$ being $i_{\iota_1}, \dots, i_{\iota_{n-k}}$

Make the replacement:

$$\mathbb{E}\,\mathrm{tr}_N (V_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots V_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}}) = \tau(v_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}}) + \frac{\nu_1(v_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}})}{N^2} + \frac{\nu_2^{(N)}(v_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}})}{N^4}.$$

$\Longrightarrow$ you get 3 convergent series.

For $|z|$ large, a series expansion to decouple U and V

$|z| > \|\xi\| + 2\sum_{i=1}^{r_1}\|\gamma_i\| + 2\sum_{i=1}^{r_2}\|\beta_i\|$, then

$$\mathbb{E}\,\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \tau\Big((zI_m - \xi)\otimes I_N \otimes 1_{\mathcal{A}} - 2\Re\sum_{i=1}^{r_1}\gamma_i \otimes U_i \otimes 1_{\mathcal{A}} - 2\Re\sum_{i=1}^{r_2}\beta_i \otimes I_N \otimes v_i\Big)^{-1}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{0 \le i_1,\ldots,i_n \le r_2}\sum_{\epsilon_{\iota_1},\ldots,\epsilon_{\iota_{n-k}} \in \{\pm 1\}} \frac{\left\{\mathbb{E}\,\mathrm{tr}_m \otimes \mathrm{tr}_N(\tilde{T}_{i_1}\cdots\tilde{T}_{i_n})\right\}\tau\left(v_{i_{\iota_1}}^{\epsilon_{\iota_1}}\cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}}\right)}{z^{n+1}}.$$

k of the i's being equal to 0,

and the other $n-k$ being $i_{\iota_1}, \ldots, i_{\iota_{n-k}}$

Make the replacement:

$$\mathbb{E}\,\mathrm{tr}_N(V_{i_{\iota_1}}^{\epsilon_{\iota_1}}\cdots V_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}}) = \tau(v_{i_{\iota_1}}^{\epsilon_{\iota_1}}\cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}}) + \frac{\nu_1(v_{i_{\iota_1}}^{\epsilon_{\iota_1}}\cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}})}{N^2} + \frac{\nu_2^{(N)}(v_{i_{\iota_1}}^{\epsilon_{\iota_1}}\cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}})}{N^4}.$$

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For $|z|$ large, a series expansion to decouple U and V

$|z| > \|\xi\| + 2\sum_{i=1}^{r_1}\|\gamma_i\| + 2\sum_{i=1}^{r_2}\|\beta_i\|$, then

$$\mathbb{E}\,\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \nu_1 \Big((zI_m - \xi)\otimes I_N \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i \Big)^{-1}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{0 \le i_1, \ldots, i_n \le r_2} \sum_{\epsilon_{\iota_1}, \ldots, \epsilon_{\iota_{n-k}} \in \{\pm 1\}} \frac{\Big\{ \mathbb{E}\,\mathrm{tr}_m \otimes \mathrm{tr}_N (\tilde{T}_{i_1} \cdots \tilde{T}_{i_n}) \Big\} \nu_1 \Big(v_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}} \Big)}{z^{n+1}}.$$

k of the i's being equal to 0,

and the other $n-k$ being $i_{\iota_1}, \ldots, i_{\iota_{n-k}}$

Make the replacement:

$$\mathbb{E}\,\mathrm{tr}_N (V_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots V_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}}) = \tau (v_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}}) + \frac{\nu_1 (v_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}})}{N^2} + \frac{\nu_2^{(N)} (v_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}})}{N^4}.$$

$\Longrightarrow$ you get 3 convergent series.

For $|z|$ large, a series expansion to decouple U and V

$|z| > \|\xi\| + 2\sum_{i=1}^{r_1}\|\gamma_i\| + 2\sum_{i=1}^{r_2}\|\beta_i\|$, then

$$\mathbb{E}\,\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \nu_2^{(N)} \left((zI_m - \xi) \otimes I_N \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i \right)^{-1}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{0 \le i_1, \ldots, i_n \le r_2} \sum_{\epsilon_{\iota_1}, \ldots, \epsilon_{\iota_{n-k}} \in \{\pm 1\}} \frac{\left\{ \mathbb{E}\,\mathrm{tr}_m \otimes \mathrm{tr}_N (\tilde{T}_{i_1} \cdots \tilde{T}_{i_n}) \right\} \nu_2^{(N)} \left(v_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}} \right)}{z^{n+1}}$$

k of the i's being equal to 0,

and the other $n-k$ being $i_{\iota_1}, \ldots, i_{\iota_{n-k}}$

Make the replacement:

$$\mathbb{E}\,\mathrm{tr}_N (V_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots V_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}}) = \tau (v_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}}) + \frac{\nu_1 (v_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}})}{N^2} + \frac{\nu_2^{(N)} (v_{i_{\iota_1}}^{\epsilon_{\iota_1}} \cdots v_{i_{\iota_{n-k}}}^{\epsilon_{\iota_{n-k}}})}{N^4}.$$

$\Longrightarrow$ you get 3 convergent series.

Thus, we obtain that for large $z \in \mathbb{C} \setminus \mathbb{R}$,

$$g_N(z) = \mathbb{E}(\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \mathrm{tr}_N)[(zI_m \otimes I_N \otimes I_N - S_N)^{-1}]$$

$$= \mathbb{E}(\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \tau)[(b \otimes I_N \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i)^{-1}]$$

$$+ \frac{1}{N^2} \mathbb{E}(\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \nu_1)[(b \otimes I_N \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i)^{-1}$$

$$+ \frac{1}{N^4} \mathbb{E}(\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \nu_2^{(N)})[(b \otimes I_N \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i)^{-1}$$

Thus, we obtain that for large $z \in \mathbb{C} \setminus \mathbb{R}$,

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$$= \mathbb{E}(\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \tau)[(b \otimes I_N \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i)^{-1}]$$

$$+\frac{1}{N^2}\mathbb{E}(\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \nu_1)[(b \otimes I_N \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i)^{-1}]$$

$$+\frac{1}{N^4}\mathbb{E}(\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \nu_2^{(N)})[(b \otimes I_N \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i)^{-1}]$$

We iterate the process in order to replace now the U_i's by the u_i's and obtain that for large z,

$$g_N(z) = g(z) + \frac{E(z)}{N^2} + \frac{\Delta_N(z)}{N^4},$$

$$E(z) = [\mathrm{tr}_m \otimes \tau \otimes \nu_1 + \mathrm{tr}_m \otimes \nu_1 \otimes \tau]$$

$$\left(((zI_m - \xi) \otimes 1_{\mathcal{A}} \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_{\mathcal{A}} - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I_{\mathcal{A}} \otimes v_i)^{-1} \right)$$

$$
\begin{aligned}
&\Delta_N(z) \\
&= \mathbb{E}(\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \nu_2^{(N)}) \Bigg[\Bigg((zI_m - \xi) \otimes I_N \otimes 1_{\mathcal{A}} - \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1_{\mathcal{A}} \\
&\quad - \sum_{i=1}^{r_1} \gamma_i^* \otimes U_i^* \otimes 1_{\mathcal{A}} - \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes I_N \otimes v_i^* \Bigg)^{-1} \Bigg] \\
&\quad + (\mathrm{tr}_m \otimes \nu_2^{(N)} \otimes \tau) \Bigg[\Bigg((zI_m - \xi) \otimes 1_{\mathcal{A}} \otimes 1_{\mathcal{A}} - \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_{\mathcal{A}} \\
&\quad - \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_{\mathcal{A}} - \sum_{i=1}^{r_2} \beta_i \otimes 1_{\mathcal{A}} \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1_{\mathcal{A}} \otimes v_i^* \Bigg)^{-1} \Bigg] \\
&\quad + (\mathrm{tr}_m \otimes \nu_1^{(N)} \otimes \nu_1) \Bigg[\Bigg((zI_m - \xi) \otimes 1_{\mathcal{A}} \otimes 1_{\mathcal{A}} - \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_{\mathcal{A}} \\
&\quad - \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_{\mathcal{A}} - \sum_{i=1}^{r_2} \beta_i \otimes 1_{\mathcal{A}} \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1_{\mathcal{A}} \otimes v_i^* \Bigg)^{-1} \Bigg].
\end{aligned}
$$

Since all involved functions are analytic on $\mathbb{C} \setminus \mathbb{R}$, the identity

$$g_N(z) = g(z) + \frac{E(z)}{N^2} + \frac{\Delta_N(z)}{N^4}$$

extends to all $z \in \mathbb{C} \setminus \mathbb{R}$.