Commutators in finite free probability

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Random unitary matrices

Notation and terminology:

- $U(d)$ is the group of $d \times d$ unitary matrices
- has a unique invariant probability measure (Haar measure)
- random unitary matrix = random element of $U(d)$ according to Haar measure

Asymptotically related to free probability:

- Voiculescu 1990: for $d \times d$ matrices A and B (converging in average eigenvalue distribution as $d \to \infty$) and random $d \times d$ unitary U, A and UBU^* are asymptotically free as $d \to \infty$
- On the other hand, in this talk d is fixed.

Review of $\boxplus_d$ and $\boxtimes_d$

Notation. For a $d \times d$ matrix X, we write the characteristic polynomial as

$$c_x(X) = \sum_{k=0}^{d} x^{d-k}(-1)^k \mathrm{e}_k(X)$$

where $\mathrm{e}_k(X)$ is the k-th elementary symmetric function in the eigenvalues of X:

$$\mathrm{e}_k(X) = \sum_{1 \le i_1 < \cdots < i_k \le d} \alpha_{i_1} \cdots \alpha_{i_k}$$

where $\alpha_1, \ldots, \alpha_d$ are the eigenvalues of X.

Review of $\boxplus_d$ and $\boxtimes_d$

Marcus, Spielman, and Srivastava realized that expected characteristic polynomials of operations involving randomly rotated matrices recover some classical operations on polynomials (studied by Walsh and Szegö in the 1920s):

Theorem (MSS 2015). *For* $0 \le k \le d$,

$$\mathbb{E}_U \mathrm{e}_k(A + UBU^*) = \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} \mathrm{e}_i(A)\mathrm{e}_j(B)$$

and

$$\mathbb{E}_U \mathrm{e}_k(AUBU^*) = \frac{1}{\binom{d}{k}} \mathrm{e}_k(A)\mathrm{e}_k(B)$$

Review of $\boxplus_d$ and $\boxtimes_d$

Definition. For polynomials

$$p(x) = \sum_{k=0}^{d} x^{d-k}(-1)^k a_k \text{ and } q(x) = \sum_{k=0}^{d} x^{d-k}(-1)^k b_k$$

define

$$p(x) \boxplus_d q(x) := \sum_{k=0}^{d} x^{d-k}(-1)^k \left(\sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} a_i b_j \right)$$

and

$$p(x) \boxtimes_d q(x) := \sum_{k=0}^{d} x^{d-k}(-1)^k \left(\frac{1}{\binom{d}{k}} a_k b_k \right)$$

Review of $\boxplus_d$ and $\boxtimes_d$

Notation.

- $e_k(A \boxplus_d B)$ etc. means the k-th coefficient of $p(x) \boxplus_d q(x)$
- $\boxminus_d$ is "subtraction" with respect to $\boxplus_d$, i.e.

$$p(x) \boxminus_d q(x) = \mathbb{E}_U c_x(A - UBU^*) = \mathbb{E}_U c_x(A + U(-B)U^*)$$

Remark.

- MSS originally proved the above theorem by replacing the continuous integrals over $U(d)$ with sums over the subgroup $H(d)$ of signed permutation matrices (key words: quadrature, minor-orthogonality)
- can also directly apply Weingarten calculus to reduce to computations with characters of symmetric groups (C-Zhi 2019)

Finite free commutator

Question. What is

$$\mathbb{E}_U c_x(AUBU^* - UBU^*A)$$

in terms of $c_x(A)$ and $c_x(B)$?

Theorem (C 2022). *With $p(x) = c_x(A)$ and $q(x) = c_x(B)$,*

$$\mathbb{E}_U c_x(AUBU^* - UBU^*A)$$
$$= (p(x) \boxminus_d p(x)) \boxtimes_d (q(x) \boxminus_d q(x)) \boxtimes_d z_d(x)$$

for a certain polynomial $z_d(x)$ which does not depend on A and B.

Finite free commutator

More generally, can look at

$$\mathbb{E}_U c_x(y_1 AUBU^* + y_2 UBU^* A)$$

for some formal variables y_1 and y_2 – this is a polynomial in x, y_1, y_2

- commutator is specialization at $y_1 = 1$ and $y_2 = -1$
- anticommutator is specialization at $y_1 = 1$ and $y_2 = 1$

Remark. Can assume WLOG that A and B are diagonal: follows from invariance of Haar measure + unitary conjugation preserving $c_x(\cdot)$

- Throughout this talk, let $A = \mathrm{diag}(a_1, \ldots, a_d)$ and $B = \mathrm{diag}(b_1, \ldots, b_d)$

Plan:

1. Weingarten calculus to turn the analytic problem into a combinatorial one
2. a bit of representation theory to reduce to a problem about immanants
3. compute the immanants using Goulden-Jackson formula

Pre-integration

Lemma. *For a fixed unitary U, we have*

$$\mathrm{e}_k(y_1 AUBU^* + y_2 UBU^* A) = \sum_{\substack{S \subseteq [d] \\ |S|=k}} \sum_{\mathrm{p}: S \to [d]} \left(\prod_{i \in S} b_{\mathrm{p}(i)} \right)$$

$$\times \sum_{\sigma \in \mathrm{Sym}(S)} \mathrm{sgn}(\sigma) \prod_{i \in S} (y_1 a_i + y_2 a_{\sigma(i)}) u_{i \mathrm{p}(i)} \overline{u_{\sigma(i) \mathrm{p}(i)}}$$

for $0 \leq k \leq d$.

Idea of proof. Straightforward application of Vieta's formula

$$\mathrm{e}_k(X) = \sum_{\substack{S \subseteq [d] \\ |S|=k}} \det(X(S,S))$$

□

Representation theory

Weingarten calculus

For integrating polynomials in unitary matrix coordinates u_{ij}:

Theorem (Collins 2003). *There is a class function* $\mathrm{Wg}^{U}_{k,d}$ *on* S_k *such that*

$$\int_{U(d)} u_{\mathrm{i}(1)\mathrm{j}(1)} \cdots u_{\mathrm{i}(k)\mathrm{j}(k)} \overline{u_{\mathrm{i}'(1)\mathrm{j}'(1)}} \cdots \overline{u_{\mathrm{i}'(l)\mathrm{j}'(l)}} \, dU$$

is equal to

$$\sum_{\substack{\pi,\sigma \in S_k \\ \mathrm{i}=\mathrm{i}'\circ\pi \\ \mathrm{j}=\mathrm{j}'\circ\sigma}} \mathrm{Wg}^{U}_{k,d}(\pi^{-1}\sigma)$$

if $k = l$, *and* 0 *otherwise.*

Representation theory

Weingarten calculus

Decomposition of $\mathrm{Wg}^{U}_{k,d}$ into irreps:

Theorem (Collins 2003).

$$\mathrm{Wg}^{U}_{k,d} = \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq d}} w_\lambda \chi^\lambda \textit{ with } w_\lambda := \frac{\dim(\lambda)^2}{(k!)^2 \mathsf{s}_\lambda(1^d)}$$

Notation:

- χ^λ is the irreducible character of S_k labeled by λ
- $\dim(\lambda)$ is the dimension of the irrep labeled by λ, i.e. $\#$ of standard tableaux of shape λ
- $\mathsf{s}_\lambda(1^d)$ is a special polynomial in d (specialization of Schur polynomial), can be described in terms of the shape of λ

Post-integration

Lemma.

$$\mathbb{E}_U \mathrm{e}_k(y_1 AUBU^* + y_2 UBU^* A)$$

$$= \sum_{\lambda \vdash k} w_\lambda \sum_{\substack{S \subseteq [d] \\ |S|=k}} \sum_{\mu \vdash k}$$

$$\sum_{\sigma \in \mathrm{Sym}(S)} \left(\mathrm{sgn}(\sigma) \sum_{\substack{\pi \in P(S) \\ t(\pi)=\mu}} \sum_{\substack{\tau \in \mathrm{Sym}(S) \\ \tau \leq \pi}} \chi^\lambda(\sigma\tau) \right) \prod_{i \in S} (y_1 a_i + y_2 a_{\sigma(i)})$$

$$\frac{\ell(\mu)!}{|\mathrm{Orb}(\mu)|} \mathrm{m}_\mu(B)$$

Idea of proof. Weingarten calculus + character expansion + elementary combinatorics □

Post-integration

Key point from last slide: we are interested in expressions of the form

$$\sum_{\sigma \in S_k} \left(\mathrm{sgn}(\sigma) \sum_{\substack{\pi \in P(k) \\ t(\pi)=\mu}} \sum_{\substack{\tau \in S_k \\ \tau \leq \pi}} \chi^{\lambda}(\sigma\tau) \right) \prod_{i=1}^{k} (y_1 x_i + y_2 x_{\sigma(i)})$$

where $x_1, \ldots, x_k$ are some scalars.

- $x_1, \ldots, x_k$ are meant to range over $a_{s_1}, \ldots, a_{s_k}$ for $S \subseteq [d]$, $|S| = k$

Plan:

1. Weingarten calculus to turn the analytic problem into a combinatorial one
2. a bit of representation theory to reduce to a problem about immanants
3. compute the immanants using Goulden-Jackson formula

Representation theory

Conjugates of Young subgroups

Proposition. *For $\lambda, \mu \vdash k$, there is a constant $C_{\lambda,\mu}$ such that*

$$\sum_{\substack{\pi \in P(k) \\ t(\pi)=\mu}} \sum_{\substack{\tau \in S_k \\ \tau \leq \pi}} \chi^{\lambda}(\sigma\tau) = C_{\lambda,\mu}\chi^{\lambda}(\sigma).$$

These constants depend on the dominance ordering of partitions and related combinatorics: they are 0 in many cases, and in the non-zero cases relevant here, there is a simple formula.

Idea of proof. The sum is over conjugates of the Young subgroup S_μ, which is very well understood in the representation theory of S_k.

Immanants

Upshot: the dependence of $\mathbb{E}_U c_x(AUBU^* - UBU^*A)$ on A is only through

$$\sum_{\sigma \in \mathrm{Sym}(S)} \chi^\lambda(\sigma) \prod_{i \in S} (y_1 a_i + y_2 a_{\sigma(i)})$$

where $\lambda \vdash k$ and $S \subseteq [d]$ with $|S| = k$.

Definition. The *immanant* of a $k \times k$ matrix $X = (x_{ij})_{i,j}$, with respect to $\lambda \vdash k$, is

$$\mathrm{Imm}^\lambda(X) := \sum_{\sigma \in S_k} \chi^\lambda(\sigma) \prod_{i=1}^{k} x_{i\sigma(i)}.$$

This is a common generalization of the determinant ($\chi^{(1^k)} = \mathrm{sgn}$) and the permanent ($\chi^{(k)} = 1$).

Plan:

1. Weingarten calculus to turn the analytic problem into a combinatorial one
2. a bit of representation theory to reduce to a problem about immanants
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Immanants

Definition (Schur polynomials). For $\lambda \vdash k$, write

$$s_\lambda(\alpha_1, \ldots, \alpha_k) = \sum_{T \in \mathrm{SST}(\lambda)} \alpha_1^{\omega_1(T)} \cdots \alpha_k^{\omega_k(T)}$$

Notation:

- $\mathrm{SST}(\lambda)$ is the set of semistandard tableaux with shape λ
- "semistandard" means weakly increasing rows, strictly increasing columns
- $\omega_i(T)$ is the number of entries i in T

Example. With $\lambda = (2, 1) \vdash 3$,

1 1	1 1	1 2	1 2	1 3	1 3	2 2	2 3
2	3	2	3	2	3	3	3
$\alpha_1^2\alpha_2$	$\alpha_1^2\alpha_3$	$\alpha_1\alpha_2^2$	$\alpha_1\alpha_2\alpha_3$	$\alpha_1\alpha_2\alpha_3$	$\alpha_1\alpha_3^2$	$\alpha_2^2\alpha_3$	$\alpha_2\alpha_3^2$

Immanants

Theorem (Goulden-Jackson 1992). *Let Y be a $k \times k$ matrix and let $Z = \mathrm{diag}(z_1, \ldots, z_k)$ be a diagonal matrix of commuting formal variables. Let $\alpha_1, \ldots, \alpha_k$ be the eigenvalues of ZY. Then*

$$\mathrm{Imm}^{\lambda}(Y) = [z_1 \cdots z_k] s_{\lambda}(\alpha_1, \ldots, \alpha_k)$$

This might seem like another complicated expression (it is) but we only care about a particular matrix: let

$$Y = (y_1 x_i + y_2 x_j)_{i,j}$$

which has rank 2, i.e. ZY has only two non-zero eigenvalues α, β

Immanants

Lemma. *For non-zero α, β, we have*

$$\mathsf{s}_\lambda(\alpha, \beta, 0, \dots, 0) = \begin{cases} \alpha^k \frac{(\beta/\alpha)^{\lambda_2} - (\beta/\alpha)^{\lambda_1+1}}{1-\beta/\alpha} & \text{if } \ell(\lambda) \le 2 \\ 0 & \text{otherwise} \end{cases}$$

1	1	1
2	2	

$\alpha^3\beta^2$

1	1	2
2	2	

$\alpha^2\beta^3$

1	2	2
2	2	

$\alpha\beta^4$

2	2	2
2	2	

β^5

Immanants

Key point:

- In the case of the commutator, i.e. $y_1 = 1$ and $y_2 = -1$, it turns out that $\beta = -\alpha$
- Then the non-zero expression for $s_\lambda(\alpha_1, \ldots, \alpha_k)$ is just $(-1)^{\lambda_2}\alpha^k$, which does not depend on λ except through a sign.

So

$$\mathrm{Imm}^\lambda(x_i - x_j)_{i,j} = \begin{cases} (-1)^{\lambda_2}\mathrm{Per}(x_i - x_j)_{i,j} & \text{if } \ell(\lambda) \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Straightforward to compute

$$\mathrm{Per}(x_i - x_j)_{i,j} = \sum_{i+j=k} (-1)^i i! j! e_i(X) e_j(X)$$

Main theorem in terms of coefficients

Theorem.

$$\mathbb{E}_U \mathrm{e}_k(AUBU^* - UBU^*A) = \left(\sum_{i+j=k}(-1)^i \frac{(d-i)!(d-j)!}{d!(d-k)!}\mathrm{e}_i(A)\mathrm{e}_j(A)\right)$$
$$\left(\sum_{i+j=k}(-1)^i \frac{(d-i)!(d-j)!}{d!(d-k)!}\mathrm{e}_i(B)\mathrm{e}_j(B)\right)\frac{(d-k)!}{(d-k/2)!}(k/2)!\frac{d+1-k/2}{d+1}$$

Corollary.

$$\mathbb{E}_U c_x(AUBU^* - UBU^*A)$$
$$= (p(x) \boxminus_d p(x)) \boxtimes_d (q(x) \boxminus_d q(x)) \boxtimes z_d(x)$$

with

$$z_d(x) = \sum_{k=0}^{\lfloor d/2 \rfloor} x^{d-2k}\binom{d}{2k}(d)_k \frac{k!}{(2k)!}\frac{d+1-k}{d+1}$$

The polynomial $z_d(x)$

$$z_d(x) = \sum_{k=0}^{\lfloor d/2 \rfloor} x^{d-2k} \binom{d}{2k} (d)_k \frac{k!}{(2k)!} \frac{d+1-k}{d+1}$$

Question. Is there a polynomial $r(x)$ such that $z_d(x) = r(x) \boxminus_d r(x)$?

- Motivated by Nica-Speicher 1998, Marcus 2016, and Arizmendi-Perales 2016, I've tried letting $r(x)$ be the "finite free Poisson" polynomial $P_{\lambda,d}(x)$
  - associated Laguerre polynomial + some normalizations
  - root distribution is asymptotically Marchenko-Pastur
- it's close but not quite right: with rate $\lambda = 1 + \frac{1}{d}$, coefficient of x^{d-2k} in $P_{\lambda,d}(x) \boxminus_d P_{\lambda,d}(x)$ is

$$(-1)^k \frac{1}{d^{2k}} \binom{d}{2k} (d)_k \frac{(2k)!}{k!} \frac{d+1}{d+1-k}$$

Finite free cumulants

Arizmendi & Perales (2016) found a sequence $\kappa_n^{(d)}$ of *finite free cumulants* which

- linearize $\boxplus_d$: $\kappa_n^{(d)}(A \boxplus_d B) = \kappa_n^{(d)}(A) + \kappa_n^{(d)}(B)$
- satisfy a moment-cumulant relation:

$$m_n^{(d)}(A) = \frac{(-1)^{n-1}}{d^{n+1}(n-1)!} \sum_{\substack{\sigma,\tau \in P(n) \\ \sigma \vee \tau = 1_n}} d^{|\sigma|+|\tau|} \mu(0_n, \sigma)\mu(0_n, \tau)\kappa_\sigma^{(d)}(A)$$

- converge to free cumulants: $\kappa_n^{(d)}(A) \to \kappa_n(A)$ as $d \to \infty$

These can be seen as symmetric functions in eigenvalues, defined in terms of $\{\mathrm{e}_k(A) : 0 \leq k \leq d\}$

Finite free cumulants

Obvious question: what is

$$\mathbb{E}_U \kappa_n^{(d)}(AUBU^* - UBU^*A)$$

in terms of $\kappa_n^{(d)}(A)$ and $\kappa_n^{(d)}(B)$? Naive attempt:

- can argue similarly to A-GV-P 2021 by mimicking their use of coefficient-cumulant formulas
- I haven't been able to do much with the resulting expression

So take a step back and look at the free commutator.

Free probability

Let a, b be free and self-adjoint. For non-trivial combinatorial reasons, can assume a and b are even. Let $\alpha_n := \kappa_{2n}(a)$ and $\beta_n = \kappa_{2n}(b)$.

Theorem (Nica-Speicher 1998). *$i(ab - ba)$ is even and*

$$\kappa_{2n}(i(ab - ba)) = 2 \sum_{\substack{\pi,\sigma \in NC(n) \\ \sigma \leq K(\pi)}} \alpha_\pi \beta_\sigma$$

Idea: ab is R-diagonal and the distribution of $ab - ba$ is is encoded in the joint distribution of ab and $ba = (ab)^*$

Free probability

Potential blueprint for finite setting:

$$\begin{aligned}
&\kappa_{2n}(i(ab - ba)) \\
&= \kappa_{2n}(ab, ba, \ldots, ab, ba) + \kappa_{2n}(ba, ab, \ldots, ba, ab) \quad (R\text{-diagonal}) \\
&= 2\kappa_{2n}(ab, ba, \ldots, ab, ba) \quad (\text{tracial}) \\
&= 2 \sum_{\substack{\pi \in NC(4n) \\ \pi \vee \sqcap \cdots \sqcap = 1_{4n}}} \kappa_\pi(a, b, b, a, \ldots, a, b, b, a) \quad (\text{products as arguments}) \\
&= 2 \sum_{\substack{\pi, \sigma \in NC(2n) \\ \sigma \leq K(\pi)}} \alpha_\pi \beta_\sigma
\end{aligned}$$

Multivariate characteristic polynomial

Mirabelli (PhD thesis, 2021) has introduced a multivariate characteristic polynomial

$$c_{x,y_1,\ldots,y_m}(A_1,\ldots,A_m) = \det(xI - y_1A_1 - \cdots - y_mA_m)$$

and shown that the formula for $\boxplus_d$ extends straightforwardly to multiple variables. Open:

$$\mathbb{E}_U c_{x,y_1,y_2}(AUBU^*,(AUBU^*)^*) = ?$$

If A and B are self-adjoint, then this is the same as

$$\mathbb{E}_U c_x(y_1 AUBU^* + y_2 UBU^* A)$$

and the commutator is the specialization at $y_1 = 1$ and $y_2 = -1$.

Questions

- understand $\mathrm{Imm}^{\lambda}(y_1 x_i + y_2 x_j)_{i,j}$, towards

$$\mathbb{E}_U c_{x,y_1,y_2}(AUBU^*, (AUBU^*)^*) = ?$$

- multivariate finite free cumulants – in terms of the coefficients of

$$c_{x,y_1,\ldots,y_n}(A_1, \ldots, A_n)?$$

- finite R-diagonality?

Thanks for listening!