Spectral norm and strong freeness

March Boedihardjo
Joint work with Afonso Bandeira and Ramon van Handel

UC Berkeley Probabilistic Operator Algebra Seminar
August 2023

Gaussian series random matrix

Gaussian series:

$$\sum_k g_k A_k$$

where the g_k are i.i.d. $N(0,1)$ random variables and the A_k are deterministic $d \times d$ matrices

Examples

- GOE
- Sparse Gaussian
- Gaussian Toeplitz

Outline

Spectral norm:

- Estimate by NC Khintchine
- Main result: Remove the dim factor in many cases
- Application: Matrix Spencer Conjecture

Strong freeness:

- Main result
- Example: Sparse Gaussian
- PT conjecture

Noncommutative Khintchine inequality

Theorem (Lust-Piquard and Pisier, ≈1990)

If $Z = \sum_k g_k A_k$ is Hermitian, where the g_k are i.i.d. $N(0,1)$ random variables and the A_k are deterministic $d \times d$ matrices, then

$$\|\mathbb{E}(Z^2)\|^{\frac{1}{2}} \lesssim \mathbb{E}\|Z\| \lesssim \sqrt{\log d}\,\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}.$$

Problem

When is the $\sqrt{\log d}$ factor needed?

$$\|\mathbb{E}(Z^2)\|^{\frac{1}{2}} \lesssim \mathbb{E}\|Z\| \lesssim \sqrt{\log d}\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}.$$

Examples

(1) If $Z = \begin{bmatrix} g_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & g_d \end{bmatrix}$, then $\mathbb{E}(Z^2) = I$ and $\mathbb{E}\|Z\| \approx \sqrt{2\log d}$.

So log factor is needed.

(2) If $Z = \frac{1}{\sqrt{d}}\begin{bmatrix} g_{1,1} & \dots & g_{1,d} \\ \vdots & \ddots & \vdots \\ g_{d,1} & \dots & g_{d,d} \end{bmatrix}$ is symmetric, then $\mathbb{E}(Z^2) = I$ and $\mathbb{E}\|Z\| \approx 2$.

So log factor is not needed.

Problem

When is the $\sqrt{\log d}$ factor needed?

$$\|\mathbb{E}(Z^2)\|^{\frac{1}{2}} \lesssim \mathbb{E}\|Z\| \lesssim \sqrt{\log d}\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}.$$

Examples

(1) If $Z = \begin{bmatrix} g_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_d \end{bmatrix}$, then $\mathbb{E}(Z^2) = I$ and $\mathbb{E}\|Z\| \approx \sqrt{2\log d}$.

So log factor is needed. (Commutative)

(2) If $Z = \frac{1}{\sqrt{d}} \begin{bmatrix} g_{1,1} & \cdots & g_{1,d} \\ \vdots & \ddots & \vdots \\ g_{d,1} & \cdots & g_{d,d} \end{bmatrix}$ is symmetric, then $\mathbb{E}(Z^2) = I$ and $\mathbb{E}\|Z\| \approx 2$.

So log factor is not needed. (Very noncommutative)

Problem

When is the $\sqrt{\log d}$ factor needed?

$$\|\mathbb{E}(Z^2)\|^{\frac{1}{2}} \lesssim \mathbb{E}\|Z\| \lesssim \sqrt{\log d}\|\mathbb{E}(Z^2)\|^{\frac{1}{2}},$$

where $Z = \sum_k g_k A_k$ is Hermitian.

Quote from Joel Tropp's monograph (2015):

More commutativity leads to a logarithm, while less commutativity can sometimes result in cancelations that obliterate the logarithm. It remains a major open question to find a simple quantity, computable from the coefficients A_k, that decides whether $\mathbb{E}\|Z\|^2$ contains a dimensional factor or not.

What this really means...

Does the following type of inequality hold for all Hermitian $Z = \sum_k g_k A_k$?

$$\mathbb{E}\|Z\| \lesssim \|\mathbb{E}(Z^2)\|^{\frac{1}{2}} + \sqrt{\log d}\,\sigma_{**}(Z),$$

where $\sigma_{**}(Z)$ measures the noncommutativity of the A_k so that when there is enough noncommutativity, $\sigma_{**}(Z) \ll \frac{\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}}{\sqrt{\log d}}$.

What this really means...

Does the following holds for all Hermitian $Z = \sum_k g_k A_k$?

$$\mathbb{E}\|Z\| \lesssim \|\mathbb{E}(Z^2)\|^{\frac{1}{2}} + \sqrt{\log d}\,\sigma_{**}(Z),$$

where $\sigma_{**}(Z)$ measures the noncommutativity of the A_k so that when there is enough noncommutativity, $\sigma_{**}(Z) \ll \frac{\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}}{\sqrt{\log d}}$.

Conjecture (2015):

$\sigma_{**}(Z) = \max_{\|u\|=\|v\|=1} (\mathbb{E}|\langle Zu, v\rangle|^2)^{\frac{1}{2}}$ might work?

Conjecture

If $Z = \sum_k g_k A_k$ is Hermitian, where the g_k are i.i.d. $N(0,1)$ random variables and the A_k are deterministic $d \times d$ matrices, then

$$\mathbb{E}\|Z\| \lesssim \|\mathbb{E}(Z^2)\|^{\frac{1}{2}} + \sqrt{\log d} \max_{\|u\|=\|v\|=1} (\mathbb{E}|\langle Zu, v\rangle|^2)^{\frac{1}{2}}.$$

- Known to hold for many cases including independent entries
- If this conjecture were true, then we have

An estimate for the spectral norm

✓ easy to use

✓ covers a wide range of random matrices

✓ sharp in many cases

Conjecture

If $Z = \sum_k g_k A_k$ is Hermitian, where the g_k are i.i.d. $N(0,1)$ random variables and the A_k are deterministic $d \times d$ matrices, then

$$\mathbb{E}\|Z\| \lesssim \|\mathbb{E}(Z^2)\|^{\frac{1}{2}} + \sqrt{\log d} \max_{\|u\|=\|v\|=1} (\mathbb{E}|\langle Zu, v\rangle|^2)^{\frac{1}{2}}.$$

- Known to hold for many cases including independent entries

- If this conjecture were true, then we have

  An estimate for the spectral norm
  ✓ easy to use
  ✓ covers a wide range of random matrices
  ✓ sharp in many cases including i.i.d. entries

- Bandeira, B., van Handel: This conjecture is not true.

Conjecture disproved

Theorem (Bandeira, B., van Handel)

If $\sigma_{**}(\,)$ *satisfy*

(1) $\sigma_{**}(Z_1 + Z_2) \leq C(\sigma_{**}(Z_1) + \sigma_{**}(Z_2))$

(2) $\sigma_{**}(UZU^*) = \sigma_{**}(Z)$ *where* U *is a deterministic unitary*

(3) $\sigma_{**}(Z \otimes I) = \sigma_{**}(Z)$

(4) $\sigma_{**}\left(\frac{1}{\sqrt{d}}\begin{bmatrix} g_{1,1} & \cdots & g_{1,d} \\ \vdots & \ddots & \vdots \\ g_{d,1} & \cdots & g_{d,d}\end{bmatrix}\right) = o((\log d)^{-\beta})$

then there is some Hermitian $Z = \sum_k g_k A_k$ *that breaks*

$$\mathbb{E}\|Z\| \lesssim \|\mathbb{E}(Z^2)\|^{\frac{1}{2}} + (\log d)^{\beta}\,\sigma_{**}(Z).$$

Example

$\sigma_{**}(Z) = \max_{\|u\|=\|v\|=1} (\mathbb{E}|\langle Zu, v\rangle|^2)^{\frac{1}{2}}$ satisfies (1)-(4).

Affirmative side: An important first step

Theorem (Tropp 2015)

For every $d \times d$ Hermitian $Z = \sum_k g_k A_k$,

$$\mathbb{E}\|Z\| \lesssim (\log d)^{\frac{1}{4}} \|\mathbb{E}(Z^2)\|^{\frac{1}{2}} + \sqrt{\log d}\, w(Z),$$

where

$$w(Z) = \sup_{Q,U,V} \|\mathbb{E} Z_1 Q Z_2 U Z_1 V Z_2\|^{1/4},$$

Z_1, Z_2 are independent copies of Z and the sup is over all unitary Q, U, V

Example

If $Z = \frac{1}{\sqrt{d}} \begin{bmatrix} g_{1,1} & \cdots & g_{1,d} \\ \vdots & \ddots & \vdots \\ g_{d,1} & \cdots & g_{d,d} \end{bmatrix}$, this gives $\mathbb{E}\|Z\| \lesssim (\log d)^{\frac{1}{4}}$.

Correct estimate: $\mathbb{E}\|Z\| \approx 2$

Tropp's quantity

$$w(Z) = \sup_{Q,U,V} \|\mathbb{E} Z_1 Q Z_2 U Z_1 V Z_2\|^{1/4}$$

measures noncommutativity in the following sense:

- Always $w(Z) \leq \|\mathbb{E}(Z^2)\|^{\frac{1}{2}}$
- If $Z = \sum_k g_k A_k$ with all the A_k commuting, then $w(Z) = \|\mathbb{E}(Z^2)\|^{\frac{1}{2}}$
- If $Z = \begin{bmatrix} g_{1,1} & \cdots & g_{1,d} \\ \vdots & \ddots & \vdots \\ g_{d,1} & \cdots & g_{d,d} \end{bmatrix}$ then $w(Z) \sim d^{\frac{1}{4}}$ and $\|\mathbb{E} Z^2\|^{\frac{1}{2}} \sim \sqrt{d}$.

Theorem (Tropp 2015)

For every $d \times d$ Hermitian $Z = \sum_k g_k A_k$,

$$\mathbb{E}\|Z\| \lesssim (\log d)^{\frac{1}{4}} \|\mathbb{E}(Z^2)\|^{\frac{1}{2}} + \sqrt{\log d}\, w(Z).$$

- Not sharp due to the $(\log d)^{\frac{1}{4}}$ factor.
- $w(Z)$ is very difficult to compute.
  Bandeira, B., van Handel:

$$w(Z) \leq \sqrt{\|\mathrm{Cov}(Z)\|^{\frac{1}{2}} \|\mathbb{E}(Z^2)\|^{\frac{1}{2}}}.$$

Main result: Spectral norm

Theorem (Bandeira, B., van Handel, Inv. Math. 2023)

For every $d \times d$ Hermitian $Z = \sum_k g_k A_k$,

$$\boxed{\mathbb{E}\|Z\| \le 2\|\mathbb{E}(Z^2)\|^{\frac{1}{2}} + C(\log d)^{\frac{3}{4}} \sqrt{\|\mathrm{Cov}(Z)\|^{\frac{1}{2}} \|\mathbb{E}(Z^2)\|^{\frac{1}{2}}}.}$$

Example

If $Z = \begin{bmatrix} g_{1,1} & \cdots & g_{1,d} \\ \vdots & \ddots & \vdots \\ g_{d,1} & \cdots & g_{d,d} \end{bmatrix}$ then this gives

$$\mathbb{E}\|Z\| \le 2\sqrt{d} + C(\log d)^{\frac{3}{4}} d^{\frac{1}{4}}.$$

Correct estimate: $\mathbb{E}\|Z\| \approx 2\sqrt{d}$.

More general setting

Theorem (Bandeira, B., van Handel)

Let $Z_1, \ldots, Z_n$ be independent $d \times d$ symmetric random matrices with $\mathbb{E} Z_i = 0$. Let $Z = \sum_{i=1}^n Z_i$. Then

$$\mathbb{E}\|Z\| \lesssim \|\mathbb{E}(Z^2)\|^{\frac{1}{2}} + (\log d)^{\frac{3}{2}} \|\mathrm{Cov}(Z)\|^{\frac{1}{2}} + (\log d)^2 (\mathbb{E} \max_i \|Z_i\|_F^2)^{\frac{1}{2}}.$$

Sharp for

- very sparse matrices
- many patterned matrices
- some block models

Matrix Spencer Conjecture

If $A_1, \ldots, A_n \in \mathbb{R}^{n \times n}$ with $\|A_i\| \leq 1$, then $\exists \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$ s.t.

$$\left\| \sum_{i=1}^{n} \epsilon_i A_i \right\| \leq C\sqrt{n}.$$

Can be thought of as: How small can the following quantity be?

$$\inf_{S \subset \{1,\ldots,n\}} \left\| \sum_{i \in S} A_i - \sum_{i \in S^c} A_i \right\|$$

Matrix Spencer Conjecture

If $A_1, \ldots, A_n \in \mathbb{R}^{n\times n}$ with $\|A_i\| \le 1$, then $\exists \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$ s.t.

$$\left\| \sum_{i=1}^{n} \epsilon_i A_i \right\| \le C\sqrt{n}.$$

Note: $C\sqrt{n}$ is sharp even for rank 1, e.g., $A_i = e_1^T e_i$

Notable consequence of our result

Matrix Spencer Conjecture

If $A_1, \dots, A_n \in \mathbb{R}^{n\times n}$ with $\|A_i\| \le 1$, then $\exists \epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$ s.t.

$$\left\| \sum_{i=1}^{n} \epsilon_i A_i \right\| \le C\sqrt{n}.$$

Theorem (Bansal, Jiang, Meka, STOC 2023)

If $A_1, \dots, A_n \in \mathbb{R}^{n\times n}$ with $\|A_i\| \le 1$ and $\mathrm{rank}(A_i) \le n/\log^3 n$, then $\exists \epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$ s.t.

$$\left\| \sum_{i=1}^{n} \epsilon_i A_i \right\| \le C\sqrt{n}.$$

Strong freeness

Let X_d and Y_d be $d \times d$ symmetric random matrices.

Asymptotic freenes:

$$\lim_{d\to\infty} \frac{1}{d} \mathrm{Tr}\, P(X_d, Y_d) = \tau(P(a,b)) \text{ a.s.},$$

where a and b are freely independent variables.

Strong asymptotic freeness: In addition,

$$\lim_{d\to\infty} \|P(X_d, Y_d)\| = \|P(a,b)\| \text{ a.s.}.$$

GUE matrices

Suppose $X_d = \frac{1}{\sqrt{d}} \begin{bmatrix} g_{1,1} & \cdots & g_{1,d} \\ \vdots & \ddots & \vdots \\ g_{d,1} & \cdots & g_{d,d} \end{bmatrix}$ and Y_d is an independent copy of X_d.

Voiculescu 1991:

$$\lim_{d \to \infty} \frac{1}{d} \mathrm{Tr}\, P(X_d, Y_d) = \tau(P(a, b)) \ a.s.,$$

where a and b are freely independent semicircular variables.

Haagerup-Thorbjørsen 2005:

$$\lim_{d \to \infty} \|P(X_d, Y_d)\| = \|P(a, b)\| \ a.s..$$

Consequence: $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ is not a group.

Strong asymptotic freeness holds for

- Wigner and deterministic (Belinschi, Capitaine 2016)
  Prior work by Schultz, Capitaine, Donati-Martin, Anderson, Male
- Haar unitary and deterministic (Collins, Male 2014)
- Random permutations (Bordenave, Collins 2019)

Main result: Strong freeness

Theorem (Bandeira, B., van Handel)
Let $X_d^{(1)}, \dots, X_d^{(r)}$ *be* $d \times d$ *self-adjoint random matrices with jointly Gaussian entries,* $\mathbb{E}X_d^{(i)} = 0$, $\mathbb{E}(X_d^{(i)})^2 = I$.

(1) *If* $\|\mathrm{Cov}(X_d^{(i)})\| = o(1)$ *as* $d \to \infty$ *then*

$$\lim_{d\to\infty} \mathbb{E}\mathrm{tr}\, P(X_d^{(1)}, \dots, X_d^{(r)}) = \tau(P(s_1, \dots, s_r)),$$

where $s_1, \dots, s_r$ *are free semicircular variables.*

(2) *If* $\|\mathrm{Cov}(X_d^{(i)})\| = o((\log d)^{-3})$ *as* $d \to \infty$ *then*

$$\lim_{d\to\infty} \|P(X_d^{(1)}, \dots, X_d^{(r)})\| = \|P(s_1, \dots, s_r)\| \quad a.s.,$$

where $s_1, \dots, s_r$ *are free semicircular variables.*

Sparse matrices

For $d \in \mathbb{N}$, let G_d be a m_d-regular graph on d vertices.
Let X_d be $d \times d$ with

$$X_d(i,j) = \begin{cases} \frac{1}{\sqrt{m_d}} g_{i,j}, & (i,j) \in \mathrm{Edge}(G_d) \\ 0, & \text{Otherwise} \end{cases}.$$

Bandeira, B., van Handel:
If $m_d \gg (\log d)^3$, then i.i.d. copies of X_d are strongly asymptotically free.

Previously not even known for any $m_d = o(d)$.

Peterson-Thom conjecture

Theorem (Bandeira, B., van Handel)
If $n(d) = o(d/(\log d)^3)$ *then*

$$\lim_{d\to\infty} \|P(X_d^{(1)} \otimes I_{n(d)}, \dots, X_d^{(r)} \otimes I_{n(d)}, I_d \otimes Y_{n(d)}^{(1)}, \dots, I_d \otimes Y_{n(d)}^{(r)})\|$$
$$= \|P(s_1 \otimes 1, \dots, s_r \otimes 1, 1 \otimes s_1, \dots, 1 \otimes s_r)\| \quad a.s.$$

Ben Hayes: PT conjecture is true if the above holds for $n(d) = d$

Belinschi, Capitaine proposed the first proof of this

Alternative proof by Bordenave, Collins

Classical vs. Free

Noncommutative Khintchine: If $Z = \sum_k g_k A_k$ is Hermitian, where the g_k are i.i.d. $N(0,1)$ random variables and the A_k are deterministic $d \times d$ matrices, then

$$\|\mathbb{E}(Z^2)\|^{\frac{1}{2}} \lesssim \mathbb{E}\|Z\| \lesssim \sqrt{\log d}\,\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}.$$

Free Khintchine: If $Z = \sum_k s_k \otimes A_k$, where the s_k are free semicircular variables, then

$$\|\mathbb{E}(Z^2)\|^{\frac{1}{2}} \le \|Z_{\mathrm{free}}\| \le 2\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}.$$

Main result: Spectral norm

Theorem (Bandeira, B., van Handel)

For every $d \times d$ Hermitian $Z = \sum_k g_k A_k$,

$$\boxed{\mathbb{E}\|Z\| \leq 2\|\mathbb{E}(Z^2)\|^{\frac{1}{2}} + C(\log d)^{\frac{3}{4}}\sqrt{\|\mathrm{Cov}(Z)\|^{\frac{1}{2}}\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}}.}$$

The proof uses free probability:

(1) Let $Z_{\mathrm{free}} = \sum_k s_k \otimes A_k$.

(2) Interpolate between Z and Z_{free}.

(3) $\|Z_{\mathrm{free}}\| \leq 2\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}$ and
the discrepancy from interpolation gives
$C(\log d)^{\frac{3}{4}}\sqrt{\|\mathrm{Cov}(Z)\|^{\frac{1}{2}}\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}}$.

Interpolation

For $0 \leq t \leq 1$, let

$$Z_t^{(N)} = \sqrt{t} \begin{bmatrix} Z_1 & & \\ & \ddots & \\ & & Z_N \end{bmatrix} + \sqrt{1-t} \frac{1}{\sqrt{N}} \begin{bmatrix} Z_{1,1} & \cdots & Z_{1,N} \\ \vdots & \ddots & \vdots \\ Z_{N,1} & \cdots & Z_{N,N} \end{bmatrix}.$$

For each N,

$$\begin{bmatrix} Z_1 & & \\ & \ddots & \\ & & Z_N \end{bmatrix} \overset{\text{dist.}}{\sim} Z$$

As $N \to \infty$,

$$\frac{1}{\sqrt{N}} \begin{bmatrix} Z_{1,1} & \cdots & Z_{1,N} \\ \vdots & \ddots & \vdots \\ Z_{N,1} & \cdots & Z_{N,N} \end{bmatrix} \overset{\text{dist.}}{\to} Z_{\text{free}}.$$

Interpolation

Thus, it suffices to bound

$$\mathbb{E}\mathrm{Tr}(Z_1^{(N)})^p - \mathbb{E}\mathrm{Tr}(Z_0^{(N)})^p = \int_0^1 \frac{d}{dt}\mathbb{E}\mathrm{Tr}(Z_t^{(N)})^p \, dt$$

This can bounded by using Gaussian interpolation on $\mathbb{E}\mathrm{Tr}(Z_t^{(N)})^p$.

Gaussian interpolation: If W and Y are independent centered Gaussian vectors in $\mathbb{R}^m$ with independent entries and $f : \mathbb{R}^m \to \mathbb{R}$, then

$$\begin{aligned}&\frac{d}{dt}\mathbb{E}f(\sqrt{t}W + \sqrt{1-t}Y)\\ =&\frac{1}{2}\sum_{i=1}^{m}(\mathrm{Var}(W_i) - \mathrm{Var}(Y_i))\mathbb{E}\frac{\partial^2 f}{\partial x_i^2}(\sqrt{t}W + \sqrt{1-t}Y).\end{aligned}$$

Interpolation

The following term from Gaussian interpolation

$$\mathbb{E}\frac{\partial^2 f}{\partial x_i^2}(\sqrt{t}W + \sqrt{1-t}Y)$$

yields something like the following:

$$\mathbb{E}\mathrm{Tr}(A_i(\ldots)A_i(\ldots)),$$

where the two $(\ldots)$ are correlated random matrices.

Apply Gaussian covariance identity. This yields something like the following.

$$\mathrm{Tr}(A_i(\ldots)A_j(\ldots)A_i(\ldots)A_j(\ldots)).$$

Interpolation

Since

$$w(Z) = \sup_{Q_1, U, V} \left\| \sum_{i,j=1}^{n} A_i Q A_j U A_i V A_j \right\|^{\frac{1}{4}} \leq \sqrt{\|\mathrm{Cov}(Z)\|^{\frac{1}{2}} \|\mathbb{E}(Z^2)\|^{\frac{1}{2}}},$$

it follows that

$$\left| \frac{d}{dt} \mathbb{E}\mathrm{Tr}(Z_t^{(N)})^p \right| \leq Cp^4 \|\mathrm{Cov}(Z)\| \|\mathbb{E}(Z^2)\| \mathbb{E}\mathrm{Tr}(Z_t^{(N)})^{p-4}.$$

Solving this differential inequality gives

$$|[\mathbb{E}\mathrm{tr}(Z_1^{(N)})^{2p}]^{\frac{1}{2p}} - [\mathbb{E}\mathrm{tr}(Z_0^{(N)})^{2p}]^{\frac{1}{2p}}| \leq Cp^{\frac{3}{4}} \sqrt{\|\mathrm{Cov}(Z)\|^{\frac{1}{2}} \|\mathbb{E}(Z^2)\|^{\frac{1}{2}}}.$$

Main result: Spectral norm

Theorem (Bandeira, B., van Handel)

For every $d \times d$ Hermitian $Z = \sum_k g_k A_k$,

$$\mathbb{E}\|Z\| \leq 2\|\mathbb{E}(Z^2)\|^{\frac{1}{2}} + C(\log d)^{\frac{3}{4}}\sqrt{\|\mathrm{Cov}(Z)\|^{\frac{1}{2}}\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}}.$$

The proof uses free probability:

(1) Let $Z_{\mathrm{free}} = \sum_k s_k \otimes A_k$.

(2) Interpolate between Z and Z_{free}.

(3) $\|Z_{\mathrm{free}}\| \leq 2\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}$ and
the discrepancy from interpolation gives
$C(\log d)^{\frac{3}{4}}\sqrt{\|\mathrm{Cov}(Z)\|^{\frac{1}{2}}\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}}$.

Extending to strong freeness

- Apply Gaussian interpolation to $\mathbb{E}\mathrm{Tr}\,|\lambda I - Z_t^{(N)}|^{-2p}$ rather than to $\mathbb{E}\mathrm{Tr}(Z_t^{(N)})^{2p}$.

- This gives not only

$$\mathbb{E}\|Z\| \leq \|Z_{\mathrm{free}}\| + C(\log d)^{\frac{3}{4}}\sqrt{\|\mathrm{Cov}(Z)\|^{\frac{1}{2}}\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}},$$

but also

$$\mathrm{sp}(Z) \subset \mathrm{sp}(Z_{\mathrm{free}}) + C(\log d)^{\frac{3}{4}}\sqrt{\|\mathrm{Cov}(Z)\|^{\frac{1}{2}}\|\mathbb{E}(Z^2)\|^{\frac{1}{2}}}[-1,1],$$

with high probability.

- By Haagerup-Thorbjørsen's linearization trick,
bounding the spectrum gives strong asymptotic freeness.

(1) Combinatorial proof:
The spectral norm of Gaussian matrices with correlated entries
A. S. Bandeira and M. T. Boedihardjo
arxiv 2104.02662 (2021)

(2) Analytic proof:
Matrix Concentration Inequalities and Free Probability
A. S. Bandeira, M. T. Boedihardjo and R. van Handel
Inventiones Mathematicae (2023)

(3) Matrix Spencer:
Resolving Matrix Spencer Conjecture Up to Poly-log Rank
N. Bansal, H. Jiang, R. Meka
STOC (2023)