Strong freeness of Gaussian matrices with correlated entries

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Norm of Gaussian matrix

Let

$$X = \begin{bmatrix} g_{1,1} & \cdots & g_{1,d} \\ \vdots & \ddots & \vdots \\ g_{d,1} & \cdots & g_{d,d} \end{bmatrix},$$

where all entries are i.i.d. $N(0,1)$ random variables.

Classical fact:

$$\mathbb{E}\|X\| \sim \sqrt{d},$$

where $\|X\| = \sup_{\|v\|_2=1} \|Xv\|_2$.

Norm of diagonal Gaussian matrix

Let

$$X = \begin{bmatrix} g_1 & & \\ & \ddots & \\ & & g_d \end{bmatrix},$$

where the diagonal entries are i.i.d $N(0,1)$ random variables and off-diagonal entries are 0.

Then

$$\mathbb{E}\|X\| = \mathbb{E} \sup_{1 \le i \le d} |g_i| \sim \sqrt{\log d}.$$

Gaussian matrix in general

Suppose that X is a $d \times d$ matrix such that
$\mathbb{E}X = 0$ and the entries are jointly Gaussian.

Lower bound:

$$\mathbb{E}\|X\| \sim (\mathbb{E}\|X\|^2)^{\frac{1}{2}} \geq \max(\|\mathbb{E}(X^*X)\|^{\frac{1}{2}}, \|\mathbb{E}(XX^*)\|^{\frac{1}{2}}).$$

Upper bound (noncommutative Khintchine):

$$\mathbb{E}\|X\| \lesssim \sqrt{\log d}\max(\|\mathbb{E}(X^*X)\|^{\frac{1}{2}}, \|\mathbb{E}(XX^*)\|^{\frac{1}{2}}).$$

$\sqrt{\log d}$ is sharp for diagonal Gaussian.
$\sqrt{\log d}$ is not needed when all d^2 entries are i.i.d. $N(0,1)$.

Question

Noncommutative Khintchine:

$$\mathbb{E}\|X\| \lesssim \sqrt{\log d}\, \max(\|\mathbb{E}(X^*X)\|^{\frac{1}{2}}, \|\mathbb{E}(XX^*)\|^{\frac{1}{2}}).$$

Question: When can the $\sqrt{\log d}$ factor be removed?

If the $\sqrt{\log d}$ factor can be removed, then

$$\mathbb{E}\|X\| \sim \max(\|\mathbb{E}(X^*X)\|^{\frac{1}{2}}, \|\mathbb{E}(XX^*)\|^{\frac{1}{2}}).$$

Very little is known in the general setting concerning the extent to which $\sqrt{\log d}$ can be removed.

Tropp 2018: Suppose that X is a $d \times d$ self-adjoint matrix with jointly Gaussian entries and $\mathbb{E}X = 0$. Then

$$\mathbb{E}\|X\| \lesssim (\log d)^{\frac{1}{4}} \|\mathbb{E}(X^2)\|^{\frac{1}{2}} + (\log d)^{\frac{1}{2}} w(X),$$

where

$$w(X) = \sup_{\substack{Q_1, Q_2, Q_3 \\ \text{unitary}}} \|\mathbb{E}(XQ_1YQ_2XQ_3Y)\|^{\frac{1}{4}},$$

and Y is an independent copy of X.

Issues: (1) $(\log d)^{\frac{1}{4}}$ factor
(2) $w(X)$ is very hard to compute.

Effective rank

If $\mathcal{M}$ is an inner product space with $\dim \mathcal{M} < \infty$ and $T : \mathcal{M} \to \mathcal{M}$ is linear and positive semidefinite, then the *effective rank of* T is

$$\mathrm{effrk}(T) = \frac{\mathrm{Tr}(T)}{\|T\|}.$$

Note that $0 \leq \mathrm{effrk}(T) \leq \dim \mathcal{M}$.

If T is an orthogonal projection, then $\mathrm{effrk}(T) = \mathrm{rank}(T)$.

Degrees of freedom

Let X be a $d \times d$ matrix with jointly Gaussian entries and $\mathbb{E}X = 0$.
Treat X as a vector $\mathrm{vec}(X)$ of length d^2.
Then $\mathbb{E}(\mathrm{vec}(X) \otimes \mathrm{vec}(X))$ is a $d^2 \times d^2$ matrix. So

$$0 \leq \mathrm{effrk}(\mathbb{E}(\mathrm{vec}(X) \otimes \mathrm{vec}(X))) \leq d^2.$$

Intuitively, $\mathrm{effrk}(\mathbb{E}(\mathrm{vec}(X) \otimes \mathrm{vec}(X)))$ measures the degrees of freedom of X. It is easy to compute in many cases.

If X is diagonal then $\mathrm{effrk}(\mathbb{E}(\mathrm{vec}(X) \otimes \mathrm{vec}(X))) = d$.
If X is Ginibre then $\mathrm{effrk}(\mathbb{E}(\mathrm{vec}(X) \otimes \mathrm{vec}(X))) = d^2$.
If X is a band matrix, then $\mathrm{effrk}(\mathbb{E}(\mathrm{vec}(X) \otimes \mathrm{vec}(X))) \sim dW$, where W is the band width.

Main result on spectral norm

Bandeira, B., van Handel (2021):
If X is $d \times d$ matrix with jointly Gaussian entries, $\mathbb{E}X = 0$ and $\mathrm{effrk}(\mathbb{E}(\mathrm{vec}(X) \otimes \mathrm{vec}(X))) \geq d(\log d)^3$, then

$$\mathbb{E}\|X\| \sim \max(\|\mathbb{E}(X^*X)\|^{\frac{1}{2}}, \|\mathbb{E}(XX^*)\|^{\frac{1}{2}}).$$

Example: Sparse matrix

Let $S \subset \{1, \ldots, d\} \times \{1, \ldots, d\}$.
Let X be a $d \times d$ matrix s.t.
$X_{i,j}$, for $(i,j) \in S$, are i.i.d. standard Gaussian and
$X_{i,j} = 0$ for $(i,j) \notin S$.

Then $\mathrm{effrk}(\mathbb{E}(\mathrm{vec}(X) \otimes \mathrm{vec}(X))) = |S|$.
So if $|S| \geq d(\log d)^3$ then

$$\mathbb{E}\|X\| \sim \max(\|\mathbb{E}(X^*X)\|^{\frac{1}{2}}, \|\mathbb{E}(XX^*)\|^{\frac{1}{2}}).$$

But it is known that this is true if $|S| \geq d \log d$ (Bandeira-van Handel 2016).

Example: Patterned matrices

Suppose that $\{S_1, \dots, S_n\}$ is a partition of $[d] \times [d]$.
Let $g_1, \dots, g_n$ be i.i.d. standard Gaussian random variables.
Define a $d \times d$ random matrix X by

$$X_{i,j} = g_k,$$

for all $(i,j) \in S_k$ and $1 \leq k \leq n$. Then
$\mathrm{effrk}(\mathbb{E}(\mathrm{vec}(X) \otimes \mathrm{vec}(X))) = d^2 / \max_{1 \leq k \leq n} |S_k|$.

So if $\max_{1 \leq k \leq n} |S_k| \leq \frac{d}{(\log d)^3}$, then

$$\mathbb{E}\|X\| \sim \max(\|\mathbb{E}(X^*X)\|^{\frac{1}{2}}, \|\mathbb{E}(XX^*)\|^{\frac{1}{2}}),$$

Fails when $|S_1| = \ldots = |S_n| = d$, e.g., random circulant.

Strong asymptotic freeness

Haagerup and Thorbjørnsen (2005): For $d \in \mathbb{N}$, let $X_1^{(d)}, \ldots, X_r^{(d)}$ be i.i.d. $d \times d$ GUE. Then

$$\lim_{d \to \infty} \|P(X_1^{(d)}, \ldots, X_r^{(d)})\| = \|P(s_1, \ldots, s_r)\|,$$

and

$$\lim_{d \to \infty} \frac{1}{d} \mathrm{Tr}(P(X_1^{(d)}, \ldots, X_r^{(d)})) = \tau(P(s_1, \ldots, s_r)),$$

for all polynomial P, where $s_1, \ldots, s_r$ are free semicircular variables.

Strong asymptotic freeness holds for

- Wigner and deterministic (Belinschi, Capitaine 2016)
  Prior work by Schultz, Capitaine, Donati-Martin, Anderson, Male
- Haar unitary and deterministic (Collins, Male 2014)
- Random permutations (Bordenave, Collins 2019)

Main result on strong asymptotic freeness

Bandeira, B., van Handel (2021): For $d \in \mathbb{N}$, let $X_1^{(d)}, \ldots, X_r^{(d)}$ be independent $d \times d$ self-adjoint matrices with jointly Gaussian entries s.t. $\mathbb{E}(X_k^{(d)}) = 0$, $\mathbb{E}[(X_k^{(d)})^2] = I$ and $\mathrm{effrk}(\mathbb{E}(\mathrm{vec}(X_k^{(d)}) \otimes \mathrm{vec}(X_k^{(d)}))) \gg d(\log d)^3$. Then

$$\lim_{d\to\infty} \|P(X_1^{(d)}, \ldots, X_r^{(d)})\| = \|P(s_1, \ldots, s_r)\|,$$

and

$$\lim_{d\to\infty} \frac{1}{d}\mathrm{Tr}(P(X_1^{(d)}, \ldots, X_r^{(d)})) = \tau(P(s_1, \ldots, s_r)),$$

for all polynomial P, where $s_1, \ldots, s_r$ are free semicircular variables.

Example: Sparse matrices

For $d \in \mathbb{N}$, let A be the adjancency matrix of a m_d-regular graph on d vertices. Let

$$X = \sqrt{\frac{d}{m_d}} A \circ G$$

where G is a $d \times d$ GOE.

If $m_d \gg (\log d)^3$ then i.i.d. copies of X are strongly asymptotically free.

In particular, if X is a periodic Gaussian band matrix with $W \gg (\log d)^3$, then i.i.d. copies of X are strongly asymptotically free.

Subspace

Let $\mathcal{M} \subset M_d(\mathbb{R})$ be a subspace (not subalgebra).
Suppose that $T^* = T$ for all $\mathcal{M}$.

Possible: $\dim \mathcal{M} = d$ and $\mathcal{M}$ is commutative, e.g.,
$\mathcal{M} = \{\text{Diagonal matrices in } M_d(\mathbb{R})\}$.

Impossible: $\dim \mathcal{M} > d$ and $\mathcal{M}$ is commutative, since then all $T \in \mathcal{M}$ can be jointly diagonalized but $\dim \{\text{Diagonal matrices in } M_d(\mathbb{R})\} = d$.

Intuition: As $\dim \mathcal{M}$ gets larger, matrices in $\mathcal{M}$ are more noncommuting.

Question: If $\dim \mathcal{M}$ gets large enough, do we have freeness?

Gaussian on a subspace

Even if $\dim \mathcal{M} > d$,
one cannot expect *all* $T_1, T_2 \in \mathcal{M}$ to be noncommuting, e.g.,
$T_1 = T_2 = I$.
But one can expect *most* $T_1, T_2 \in \mathcal{M}$ to be noncommuting.

A standard Gaussian X on $\mathcal{M}$ is

$$X = \sum_{k=1}^{\dim \mathcal{M}} g_k A_k,$$

where $g_1, \ldots, g_{\dim \mathcal{M}}$ are i.i.d. standard Gaussian random variables
and $(A_1, \ldots, A_{\dim \mathcal{M}})$ is an orthonormal basis for $\mathcal{M}$, i.e.,
$\mathrm{Tr}(A_i A_j^*) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$.

Dimension

For $d \in \mathbb{N}$, let $\mathcal{M} \subset \{T \in M_d(\mathbb{R}) | \, T^* = T\}$. If $\dim \mathcal{M} \gg d(\log d)^3$, then two independent X_1, X_2 chosen from $\mathcal{M}$ according to (some appropriately normalized) standard Gaussian distribution are strongly asymptotically free if $\mathbb{E}(X_1^2) = \mathbb{E}(X_2^2) = I$.

If $\dim \mathcal{M} = d$, other relations can occur, e.g.,
commuting (diagonal matrices),
half-commuting (reverse circulant matrices), etc.