

**Combinatorics of the Quantum Symmetric
Simple Exclusion Process,
Schröder trees and free cumulants**

Probabilistic Operator Algebra Seminar
Berkeley/Zoom
14 february 2022

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-Encode the asymptotic fluctuations of the *Quantum Symmetric Simple Exclusion Process*.

-Goal of the talk: Give a combinatorial formula for Q_σ ; explain that they are *free cumulants*

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$$[x_i](Q_\sigma - Q_{s_i \sigma s_i})|_{x_i=x_{i+1}} + [x_{i+1}](Q_\sigma - Q_{s_i \sigma s_i})|_{x_i=x_{i+1}} = \\ 2([x_i]Q_{\sigma^-}(x^-))([x_{i+1}]Q_{\sigma^+}(x^+))$$

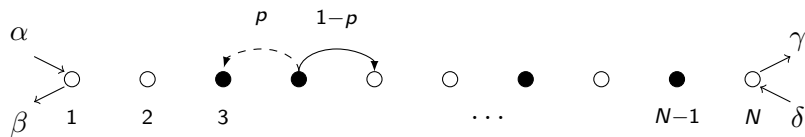
($s_i \sigma = \sigma^+ \sigma^-$, σ^+ , moves $i + 1$ and σ^- , moves i .)

Exclusion Process

The *exclusion process* describes particles hopping on an interval $\{1, 2, \dots, N\}$ and satisfying the *exclusion principle*: at most one particle per site

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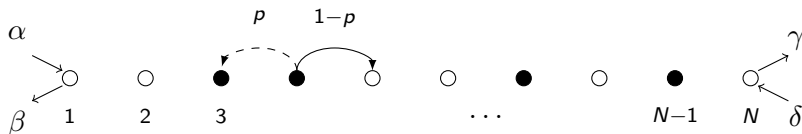
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In the large time limit a current is established and the configuration of particles converges to a *stationary measure* μ which is a probability measure on the set of configurations

$$\Omega = \{0, 1\}^N$$

Quantum Symmetric Simple Exclusion Process

Fermionic particles on $\{1, 2, \dots, N\}$ are subject to a Hamiltonian

$$H_t = \sum_{j=1}^N c_{j+1}^\dagger c_j W_t^j + c_j^\dagger c_{j+1} \bar{W}_t^j$$

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V is the quantum version of $\Omega = \{0, 1\}^N$.

A state of the form $e_{i_1} \otimes \dots \otimes e_{i_N}$ corresponds to a classical configuration (e_0 = empty, e_1 = occupied).

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It satisfies the evolution equation:

$$d\rho_t = -i[dH_t, \rho_t] - \frac{1}{2}[dH_t, [dH_t, \rho_t]] + \mathcal{L}_{bdry}(\rho_t)dt$$

\mathcal{L}_{bdry} is a boundary term describing what happens at the boundary sites $1, N$.

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ρ_t is a random matrix, if the initial configuration is diagonal on the classical states the expected value $\bar{\rho}_t$ satisfies the same evolution as the classical SSEP.

Asymptotics and loop polynomials

As $t \rightarrow \infty$ one has $\rho_t \rightarrow \rho$ in distribution
 ρ is the *stationary state* (a random $2^N \times 2^N$ matrix)

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The fluctuations of \mathbf{G} are measured by their cumulants (connected correlation functions in physics)

$$E[G_{i_1 j_1} G_{i_2 j_2} \dots G_{i_p j_p}]^c = C_p(G_{i_1 j_1}, G_{i_2 j_2}, \dots, G_{i_p j_p})$$

These are the quantities of interest.

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Only the ones for which j_1, \dots, j_p is a cyclic permutation of i_1, \dots, i_p have a nonzero limit.

If $i_1/N, i_2/N, \dots, i_p/N \rightarrow u_1, u_2, \dots, u_p \in [0, 1]$ as $N \rightarrow \infty$, then

$$E[G_{i_1 i_p} G_{i_p i_{p-1}} \cdots G_{i_2 i_1}]^c = \frac{1}{N^{p-1}} g_p(u_1, \dots, u_p) + O\left(\frac{1}{N^p}\right)$$

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The g_p are piecewise polynomial functions, polynomial in each sector corresponding to an ordering of the u_i .

Loop polynomials (Bernard and Jin, 2021)

Define $Q_\sigma(x_1, \dots, x_p)$ for $0 \leq x_1 \leq x_2 \leq \dots \leq x_p \leq 1$, indexed by circular permutations σ of $1, \dots, p$ by

$$E[G_{i_1 i_{\sigma^{p-1}(1)}} G_{i_{\sigma^{p-1}(1)} i_{\sigma^{p-2}(1)}} \dots G_{i_{\sigma(1)} i_1}]^c = \frac{1}{N^{p-1}} Q_\sigma(x_1, \dots, x_p) + O\left(\frac{1}{N^p}\right).$$

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The Q_σ are the loop polynomials. They give the values of the functions g_p in each sector.

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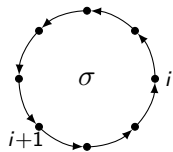
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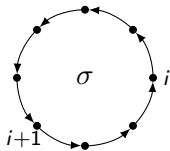
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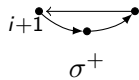
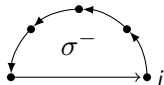
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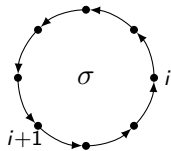
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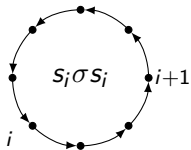
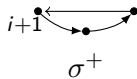
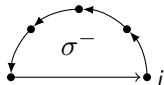


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How to use the defining relations

$$Q_\sigma = A + x_j B + x_{j+1} C + x_j x_{j+1} D,$$

$$Q_{s_j \sigma s_j} = A' + x_j B' + x_{j+1} C' + x_j x_{j+1} D',$$

where $A, B, C, D, A', B', C', D'$ do not depend on x_j, x_{j+1} .

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Let $s_i \sigma = \sigma^- \sigma^+$ and

$$\Delta := ([x_i] Q_{\sigma^-}(x^-))([x_{i+1}] Q_{\sigma^+}(x^+)).$$

By the exchange condition

$$B - C' = B' - C = \Delta.$$

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Examples: $n = 5$

$$Q_{12345} = x_1(1-4x_2-3x_3-2x_4+9x_2x_3+7x_2x_4+5x_3x_4-14x_2x_3x_4)(1-x_5)$$

$$Q_{13245} = x_1(1-6x_2-x_3-2x_4+9x_2x_3+10x_2x_4+2x_3x_4-14x_2x_3x_4)(1-x_5)$$

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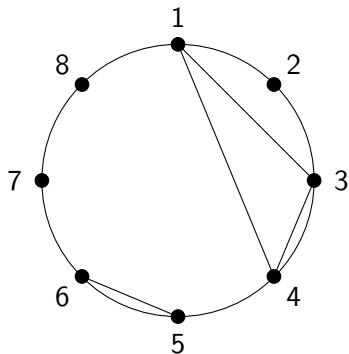
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The goal is to give an explicit combinatorial formula for the Q_σ .

Non-crossing partitions

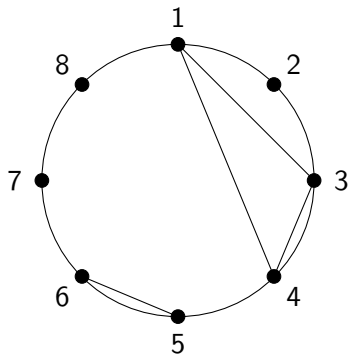
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$$|NC(n)| = \text{Cat}_n = \frac{1}{n+1} \binom{2n}{n}$$

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The *free cumulants* are $\kappa_n = n$ -linear form on A , $n = 1, 2, \dots$, defined implicitly by

$$\varphi(a_1 a_2 \dots a_n) = \sum_{\pi \in NC(n)} \kappa_{\pi}(a_1, a_2, \dots, a_n)$$

where $\kappa_{\pi}(a_1, \dots, a_n) = \prod_{\rho \text{ part of } \pi} \kappa_{|\rho|}(a_{i_1}, a_{i_2}, \dots, a_{i_{|\rho|}})$

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$$\varphi(a_1 a_2 \dots a_n) = \sum_{\pi \in NC(n)} \kappa_{\pi}(a_1, a_2, \dots, a_n)$$

$$\text{where } \kappa_{\pi}(a_1, \dots, a_n) = \prod_{\rho \text{ part of } \pi} \kappa_{|\rho|}(a_{i_1}, a_{i_2}, \dots, a_{i_{|\rho|}})$$

One has

$$\kappa_n(a_1, a_2, \dots, a_n) = \sum_{\pi \in NC(n)} \mu(\pi) \varphi_{\pi}(a_1, \dots, a_n)$$

where μ is the Möbius function of $NC(n)$:

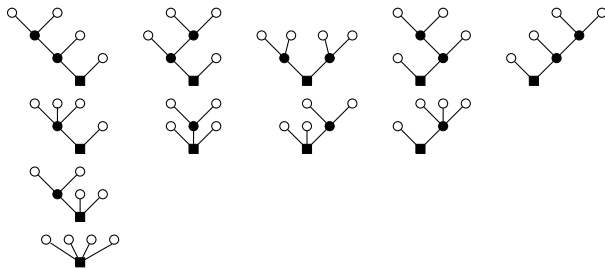
$$\mu(\pi) = \prod_{\rho \text{ part of } K(\pi)} (-1)^{|\rho|-1} \text{Cat}_{|\rho|-1}$$

Schröder trees

Plane, rooted trees such that each internal vertex has at least two descendants.

Schröder trees

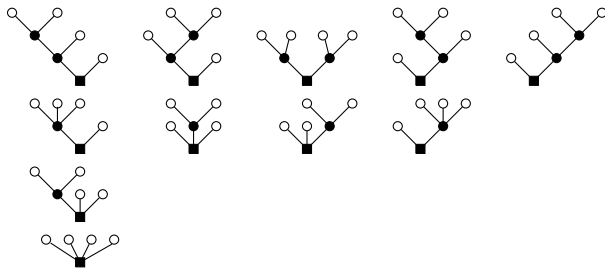
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Schröder trees with 4 leaves

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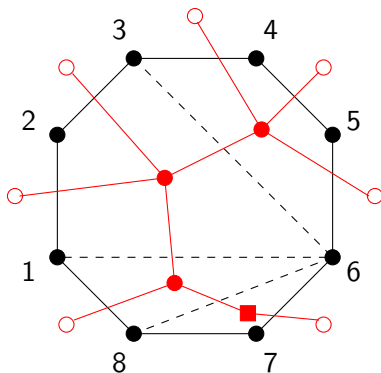
Schröder trees with 4 leaves

Counted, in terms of the number of leaves, by the small Schröder numbers $s_n = 1, 1, 3, 11, 45, \dots$ for $n = 1, 2, 3, \dots$ (A001003 in

OEIS) with generating series $\frac{1+x-\sqrt{1-6x+x^2}}{4x}$.

Schröder trees and associahedra

Schröder trees correspond to *dissections of polygons* and faces of the *associahedron*.



Prime Schröder trees

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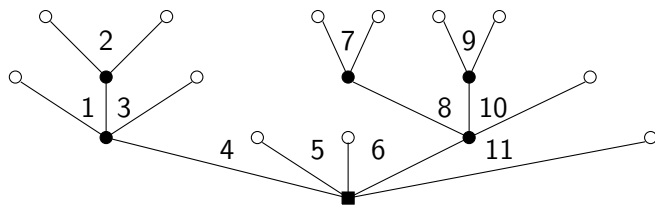
For each Schröder tree t , with $n - 1$ leaves, we can build two prime Schröder trees t_1 and t_2 , with n leaves



Corners

corner = angle between pair of consecutive edges.

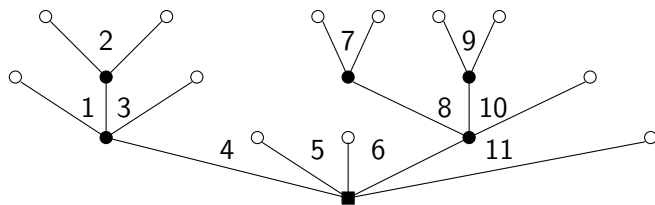
A Schröder tree with n leaves has exactly $n - 1$ corners, numbered from left to right.



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A Schröder tree determines a non-crossing partition of its corners:

$$\pi(t) = \{1, 3\}, \{2\}, \{4, 5, 6, 11\}, \{7\}, \{8, 10\}, \{9\}$$

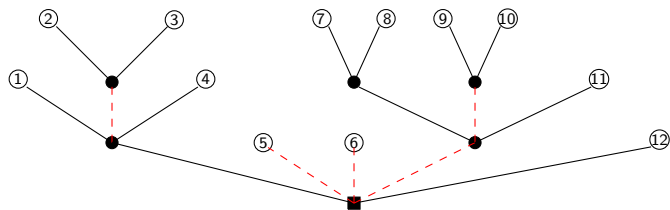
Prime Schröder trees and Möbius function on $NC(n)$

(Josuat-Vergès, Menous, Novelli, Thibon, 2017)

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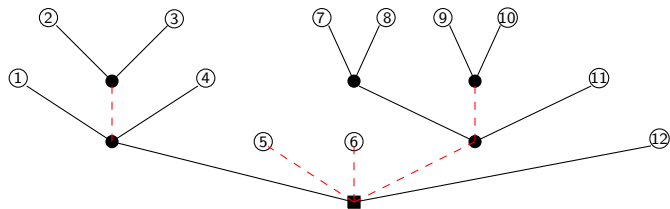
In a prime Schröder tree remove, for each internal vertex, the internal edges pointing out of this vertex, except the leftmost and rightmost ones.



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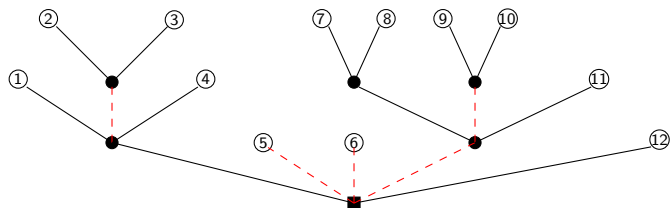


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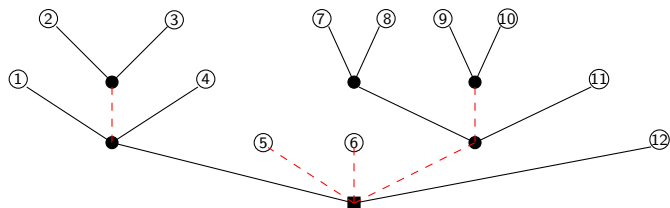
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For each $\pi \in NC(n)$ there are

$$\prod_{p \text{ part of } K(\pi)} \text{Cat}_{|p|-1} = |\mu(\pi)|$$

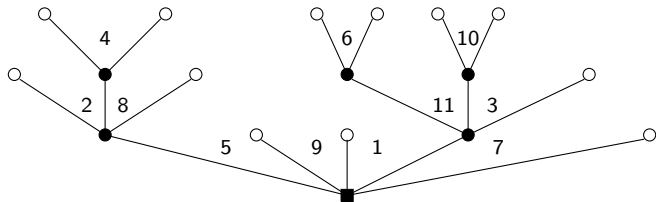
A formula for the loop polynomials

A formula for the loop polynomials

$t =$ prime Schröder tree t , with $n + 1$ leaves, $k \in [1, n]$ and σ circular permutation.

Label the corners of t , by the numbers $\sigma(k), \sigma^2(k), \dots, \sigma^{n-1}(k), k$.

$$\sigma = 2, 4, 8, 5, 9, 1, 6, 11, 10, 3, 7, \quad k = 7$$



For each internal vertex $i(v) =$ the smallest label of corners of v .

$$x^{t,k,\sigma} = \prod_v (-x_{i(v)}) \quad (\text{here } x^{t,7,\sigma} = (-1)^6 x_2 x_4 x_1 x_6 x_3 x_{10})$$

Theorem: for each $k \in [1, n]$ one has

$$Q_\sigma(x_1, \dots, x_n) = - \sum_{t \in pS_{n+1}} x^{t, k, \sigma}$$

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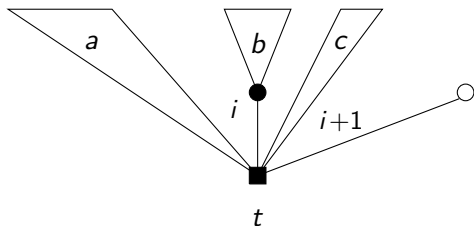
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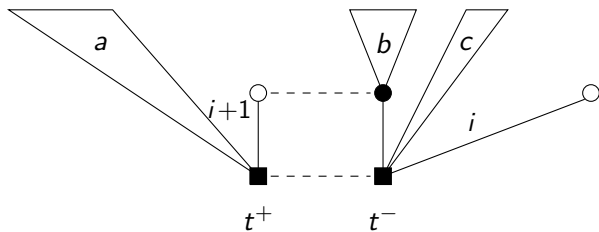
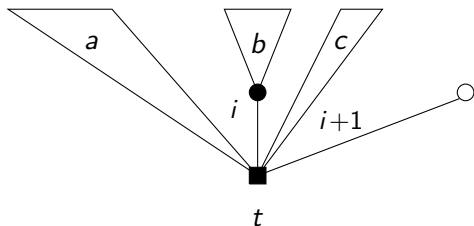
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Then check the continuity and the exchange conditions (uses simple properties of Schröder trees).

Proof of the exchange condition by cutting a prime Schröder tree into two prime Schröder trees



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On $[0, 1] \subset \mathbf{R}$ with Lebesgue measure let for $x \in [0, 1]$

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Proof: use the connection between Schröder trees, $NC(n)$ and the Möbius function.

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THANK YOU