$$\mu_{X_1+X_2}$$

Joint work with Ping Zhong

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Background

- $(\mathscr{A},\tau)$ tracial W*, $\widetilde{\mathscr{A}}$ algebra of affiliated operators
- $X \in \widetilde{\mathscr{A}} \mapsto \mu_X$ probability measure on $\mathbb{C}$
  defined in two overlapping cases

  X normal, $\mu_X(\sigma) = \tau(E_X(\sigma))$

  (L. Brown) $X \in \mathrm{Log}^+(\tau)$, that is

$$\tau(\log^+(|X|)) < +\infty.$$

- Defining property:

$$\tau(\log(|X-\lambda|)) = \int_{\mathbb{C}} \log|z-\lambda|\, d\mu_X(z)$$

$$(= \int_{[0,+\infty)} \log t\, d\mu_{|X-\lambda|}(t),\ \text{possibly } -\infty)$$

Up to a constant multiple, μ_X is the Laplacian of this subharmonic function.

Goal

- $X_1, X_2 \in \mathrm{Log}^+(\tau)$ "known" $*$-free, $X = X_1 + X_2$
  Find μ_X.
- We assume X_2 is R-diagonal (Nica-Speicher, Haagerup-Schulz)
  X_2 and UX_2 have the same $*$-distribution of U is unitary and $*$-free from X_2
- Uses
  - invariant subspaces (H-S, Dykema, et al.)
  - predictive value for eigenvalues of large rms
  - generalizes free convolution

Some *known cases*

- $X_1 = 0$ (Haagerup-Larsen-Schulz)
- $X_2 = \varepsilon XY^{-1}$, X, Y free circular (H-S)
- additional examples and insights (Biane-Lehner)
- Some Brownian motions (Hall-Kemp, H-Ho-K)
  Uses pde technique. Later reproved by free probability methods (Zhong) for $X_1 = X_1^*$.

Free convolution

- $T_1, T_2 \in \widetilde{\mathscr{A}}$ free, selfadjoint, $T = T_1 + T_2$, so $\mu_T = \mu_{T_1} \boxplus \mu_{T_2}$

$$G_T(z) = \tau((z - T)^{-1}), H_T(z) = \frac{1}{G_T(z)} - z, \quad z \in \mathbb{C}^+.$$

- Subordination when $T_j \neq \lambda$: $\exists! \omega_j : \overline{\mathbb{C}^+} \to \overline{\mathbb{C}^+} \cup \{\infty\}$ co ntinuous, analytic on $\mathbb{C}^+$ so

$$G_T(z) = G_{T_j}(\omega_j(z)), \quad \omega_1(z) + \omega_2(z) = z + \frac{1}{G_T(z)}, \quad z \in \mathbb{C}^+.$$

(History: V, B, B)

- Also (B-B-H) $\omega_1(z), z \in \overline{\mathbb{C}^+}$, (e.g., $z = 0$) is the Denjoy-Wolff point of the map

$$w \mapsto z + H_{T_2}(z + H_{T_1}(w)), \quad w \in \mathbb{C}^+.$$

Free convolution, $\mathrm{Log}^+(\tau)$

- Suppose now T_1, T_2 are free, selfadjoint and in $\mathrm{Log}^+(\tau)$

$$L_T(z) = \tau(\log(z - T)) = \int_{\mathbb{R}} \log(z-t)\, d\mu_T(t)$$

(principal branch for $z \in \mathbb{C}^+$) so $L'_T = G_T$.

- Differentiate and multiply by $G_T = G_{T_J} \circ \omega_j$

$$\omega_1(z) + \omega_2(z) = z + \frac{1}{G_T(z)} \text{ to obtain}$$

$$G_{T_1}(\omega_1(z))\omega_1'(z) + G_{T_2}(\omega_2(z))\omega_2'(z) = G_T(z) - \frac{G_T'(z)}{G_T(z)}$$

- All terms are derivatives, so

$$L_{T_1}(\omega_1(z)) + L_{T_2}(\omega_2(z)) = L_T(z) + \log\frac{1}{G_T(z)} + c$$

and examining $z = iy$, $y \to +\infty$, shows $c = 0$.

Convolution in terms of L_T

- Thus in $\mathbb{C}^+$ and even on $\mathbb{R}$ if $\omega_j(z) \in \mathbb{C}^+$

$$L_T(z) = L_{T_1}(\omega_1(z)) + L_{T_2}(\omega_2(z)) - \log(\omega_1(z) + \omega_2(z) - z)$$

- Note: if T has symmetric distribution,

$$\begin{aligned} G_T(iy) &= \frac{1}{2}\left(\int_{\mathbb{R}} \frac{d\mu_T(t)}{iy - t} + \int_{\mathbb{R}} \frac{d\mu_T(t)}{iy + t}\right) \\ &= \int_{\mathbb{R}} \frac{-iy\, d\mu_T(t)}{t^2 + y^2} = -iy\tau((T^2 + y^2)^{-1}) \end{aligned}$$

Similarly, (even for $y = 0$)

$$\Re L_T(iy) = \frac{1}{2}\tau(\log(T^2 + y^2))$$

Nonselfadjoint operators

- $X = X_1 + X_2,\ X_1, X_2 \in \mathrm{Log}^+(\tau)$ *-free
- $\widetilde{\mu}_X$ is the symmetrization of $\mu_{|X|}$
- If X_1 and X_2 are R-diagonal, we have (N-S,H-L,H-S)

$$\widetilde{\mu}_X = \widetilde{\mu}_{X_1} \boxplus \widetilde{\mu}_{X_2}$$

- In particular,

$$\tau(\log|X|) = \int_{\mathbb{R}} \log|t|\, d\widetilde{\mu}_X = \int_{\mathbb{R}} \log|t| d(\widetilde{\mu}_{X_1} \boxplus \widetilde{\mu}_{X_2})$$

General X_1

- More generally, suppose only X_2 is R-diagonal, choose a Haar unitary $*$-free from $\{X_1, X_2\}$. Then (H-S,B-L)

$$|X_1 + U^* X_2 - \lambda| = |U(X_1 - \lambda) + X_2|$$

- Since $U^* X_2 =_{\text{in distribution}} X_2$ and X_2 is free from $U(X_1 - \lambda)$,

$$\widetilde{\mu}_{|X-\lambda|} = \widetilde{\mu}_{|X_1-\lambda|} \boxplus \widetilde{\mu}_{|X_2|}$$

(Note that $m_p(\widetilde{\mu}) = m_p(\mu)$, $m_p(\mu) = \int_{\mathbb{R}} |t|^p d\mu(t), p \in \mathbb{R}$.)

$\boxplus$ *for symmetric* μ

- $\mu_1, \mu_2, \mu = \mu_1 \boxplus \mu_2$ symmetric measures on $\mathbb{R}$

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}, H_\mu(z) = \frac{1}{G_\mu(z)} - z,$$

$$G_\mu = G_{\mu_j} \circ \omega_j, \omega_1 + \omega_2 = z + \frac{1}{G_\mu(z)}.$$

- Symmetry: $G_\mu(z) = -\overline{G_\mu(-\overline{z})}$ in $\mathbb{C}^+$.
- It follows that $\omega_j(z) = -\overline{\omega_j(-\overline{z})}$, so $-i\omega_j(iy) > 0$ for $y > 0$. Also, $iG_\mu(iy) > 0$ and $-iH_\mu(iy) > 0$.

(Fact) The function

$$y \mapsto -iyH_\mu(iy)$$

increases strictly on $(0, +\infty)$ from $1/m_{-2}(\mu)$ to $m_2(\mu)$. Exception: $\mu = \frac{1}{2}(\delta_a + \delta_{-a})$. ($m_p(\mu) = \int_{\mathbb{R}} t^p \, d\mu(t)$ the moment of order p)

$\mu = \mu_1 \boxplus \mu_2$, *symmetric* μ_j

- Recall
  $L_\mu(z) = L_{\mu_1}(\omega_1(z)) + L_{\mu_2}(\omega_2(z)) - \log(\omega_1(z) + \omega_2(z) - z)$
- We will want $\Re L_\mu(0)$, so we look at $\omega_1(0)$, the Denjoy-Wolff point of $H_{\mu_2} \circ H_{\mu_1}$.
- Possibilities for the pair $(\omega_1(0), \omega_2(0))$:
  - $-i\omega_1(0) \in (0, +\infty) \Leftrightarrow -i\omega_2(0) \in (0, +\infty)$ because

$$\omega_2(0) = \frac{1}{G_{\mu_1}(\omega_1(0))} - \omega_1(0)$$

(exception: $\mu_1 = \delta_0$ so we declare it uninteresting). The fixed point equation is

$$\omega_1(0) = H_{\mu_2}(H_{\mu_1}(\omega_1(0)) = H_{\mu_2}(\omega_2(0)),$$
$$\omega_1(0)H_{\mu_1}(\omega_1(0)) = \omega_2(0)H_{\mu_2}(\omega_2(0)),$$

and it shows the intervals $J_j = (1/m_{-2}(\mu_j), m_2(\mu_j))$ intersect. (exception: $\mu_1 = (\delta_s + \delta_{-s})/2$). Also,

$$H'_{\mu_1}(\omega_1(0)) \times H'_{\mu_2}(\omega_2(0)) < 1$$

$\boxplus$ for symmetric μ

- other possibilities
  - $\omega_1(0) = \omega_2(0) = \infty$ not possible

    $$\frac{\omega_1(iy)}{\omega_2(iy)} = -1 + \frac{iy}{\omega_2(iy)} + \frac{1}{\omega_2(iy)G_{\mu_2}(\omega_2(iy))} \to 0 \text{ as } y \downarrow 0, \text{ etc.}$$

  - $\omega_1(0) = \omega_2(0) = 0$ rarely, when $\mu_1(\{0\}) + \mu_2(\{0\}) \geq 1$, in which case $G_\mu(0) = \infty$. The Julia-Carathéodory derivative of $H_{\mu_2} \circ H_{\mu_1}$ at zero is

    $$\frac{1-\mu_1(\{0\})}{\mu_1(\{0\})} \times \frac{1-\mu_2(\{0\})}{\mu_2(\{0\})} \leq 1$$

  - finally, $\omega_1(0) = 0, \omega_2(0) = \infty$. In this case $J_1 \cap J_2 = \varnothing$ and

    $$\omega_1'(0) = \frac{1/m_{-2}(\mu_1)}{(1/m_{-2}(\mu_1)) - m_2(\mu_2)},$$

    $$\lim_{y\downarrow 0}(-iy\omega_2(iy)) = (1/m_{-2}(\mu_1)) - m_2(\mu_2)$$

$X = X_1 + X_2$

- X_2 R-diagonal. As seen earlier,

$$\widetilde{\mu}_{|X-\lambda|} = \widetilde{\mu}_{|X_1-\lambda|} \boxplus \widetilde{\mu}_{X_2}$$

- Denote $\mu_1^{(\lambda)} = \widetilde{\mu}_{|X_1-\lambda|}$, $\mu_2 = \widetilde{\mu}_{X_2}$, $\mu^{(\lambda)} = \widetilde{\mu}_{|X-\lambda|} = \mu_1^{(\lambda)} \boxplus \mu_2$, $\omega_1^{(\lambda)}, \omega_2^{(\lambda)}$ the subordination functions.

$$\begin{aligned}\tau(\log|X-\lambda-z|) &= \int_{\mathbb{R}} \log|t-z|\, d\mu^{(\lambda)}(t) \\ &= \int_{\mathbb{R}} \log|t-\omega_1^{(\lambda)}(z)|\, d\mu_1^{(\lambda)}(t) \\ &+ \int_{\mathbb{R}} \log|t-\omega_2^{(\lambda)}(z)|\, d\mu_2(t) \\ &- \log|\omega_1^{(\lambda)}(z)+\omega_2^{(\lambda)}(z)-z|\end{aligned}$$

$X = X_1 + X_2$

Main case: $\omega_j(0) \notin \{0, \infty\}$, set $z = 0$:

$$\begin{aligned}
\tau(\log|X-\lambda|) &= \int_{\mathbb{R}} \log|t| \, d\mu^{(\lambda)}(t) \\
&= \int_{\mathbb{R}} \log|t - \omega_1^{(\lambda)}(0)| \, d\mu_1^{(\lambda)}(t) \\
&+ \int_{\mathbb{R}} \log|t - \omega_2^{(\lambda)}(0)| \, d\mu_2(t) \\
&- \log|\omega_1^{(\lambda)}(0) + \omega_2^{(\lambda)}(0)| \text{ or}
\end{aligned}$$

$$\begin{aligned}
\tau(\log|X-\lambda|^2) &= \tau(\log(|X_1 - \lambda|^2 + |\omega_1^{(\lambda)}(0)|^2) \\
&+ \tau(\log(|X_2|^2 + |\omega_2^{(\lambda)}(0)|^2) \\
&- \log|\omega_1^{(\lambda)}(0) + \omega_2^{(\lambda)}(0)|^2
\end{aligned}$$

$X = X_1 + X_2$

- Analysis of the fixed point of $\varphi_\lambda = H_{\mu_2} \circ H_{\mu_1^{(\lambda)}}$ shows $\omega_j^{(\lambda)}(0)$ to be differentiable.
- Further manipulation yields a density for μ_X (relative to area) at the points λ where $\omega_1(\lambda) \notin \{0, \infty\}$. To simplify, use the notation

$$K_1^{(\lambda)}(z) = zH_{\mu_1^{(\lambda)}}(z), K_2(z) = zH_{\mu_2}(z)$$

Per me si va nella città dolente. . .

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$$\frac{1}{\pi}|\omega_1^{(\lambda)}(0)|^2\tau\left[(|X_1-\lambda|^2+|\omega_1^{(\lambda)}(0)|^2)^{-1}(|(X_1-\lambda)^*|^2+|\omega_1^{(\lambda)}(0)|^2)^{-1}\right]$$

$$+\frac{1}{\pi}\frac{2|\omega_1^{(\lambda)}(0)|(|\omega_1^{(\lambda)}(0)|-|K_2'(\omega_2^{(\lambda)}(iy))|)\left|\frac{\tau((|X_1-\lambda|^2+|\omega_1^{(\lambda)}(0)|^2)^{-1}\frac{d}{d\lambda}(|X_1-\lambda|^2+|\omega_1^{(\lambda)}(0)|^2)}{\tau((|X_1-\lambda|^2+|\omega_1^{(\lambda)}(0)|^2)^{-1})^2}\right|^2}{1-\varphi_\lambda'(\omega_1^{(\lambda)}(0))}$$

- Effective calculation is possible in several cases, recovers existing results
- This gives a density on an open set $\Omega(X_1, X_2)$ whose closure (plus finitely many points) contains the support of μ_X and whose boundary has measure zero in all known cases.
- If $m_{-2}(\mu_{X_2}) = m_2(\mu_{X_2}) = \infty, \Omega(X_1, X_2) = \mathbb{C}$.

Thank you!