University of California Berkeley Probabilistic Operator Algebras Seminar

A random matrix approach to the Peterson-Thom conjecture

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We are interested in the case $G = \mathbb{F}_r$ for $r > 1$, in this case $L(G)$ is the vNa generated by r free Haar unitaries.

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Let $r > 1$ and $N \leq L(\mathbb{F}_r)$ a diffuse hyperfinite vNa. Then there exists a unique maximal amenable $P \leq L(\mathbb{F}_r)$ with $N \leq P$.

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We outline a random matrix approach.

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$$\mathbb{E}(f(X)) = \int f \, d\mu_X \text{ if } f\colon \mathbb{R} \to \mathbb{R} \text{ is bounded, Borel.}$$

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$$\ell_x\colon \mathbb{C}^*\langle T_1, \cdots, T_r\rangle \to \mathbb{C}$$

by $\ell_x(P) = \tau(P(x))$.

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For $C > 0$ the space $\Sigma_{r,C}$ corresponding to laws ℓ_x with $\|x_j\| \leq C$ is wk*-compact.

Voiculescu's Asymptotic Freeness Theorem

Theorem (Voiculescu's Asymptotic Freeness theorem)

Let $X^{(N)} = (X_1^{(N)}, \cdots, X_r^{(N)})$ *be* $N \times N$*, independent, standard GUE matrices. Then, almost surely,*

$$\ell_{X^{(N)}} \to_{N\to\infty} \ell_s$$

where $s = (s_1, \cdots, s_r)$ *is an* r*-tuple of freely independent semicirculars of mean zero and variance* 1*.*

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Generalizations due to Male, Male-Collins, Collins-Guionnet-Parraud, $\cdots$

The ⊗-HT conjecture

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Microstates Free Entropy Dimension and 1-Bounded Entropy

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where $\mathcal{O}$ is a weak*-neighborhood of ℓ_A, and $R > \max_j \|x_j\|$ is constant.

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The PT-conjecture is implied by

Conjecture (1-Bounded entropy conjecture)

If $N \leq L(\mathbb{F}_r)$ is diffuse and satisfies $h(N) = 0$, then N is hyperfinite.

How they're related

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Conjecture (Randomized Jung's Theorem conjecture)

If $N \leq M$ is not hyperfinite, then for almost every (X, Y) we have $\theta_U\big|_X$ is not unitarily conjugate to $\theta_Y\big|_N$.

$\otimes$-HT and the Randomized Jung Theorem

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$\otimes$-HT and the Randomized Jung Conjecture, II

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$$\frac{1}{l}\left\|\sum_{i=1}^{l} f_i(X^{(N)}) V f_i(Y^{(N)})^*\right\|_2 \leq C$$

$$\approx C\left\|\sum_{i=1}^{l} f_i(X^{(N)}) f_i(X^{(N)})^*\right\|_2 .$$

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Theorem

We have the following implications:

$$\otimes\text{-}HT \textit{ conjecture} \implies \textit{Randomized Jung theorem conjecture}$$

$$\implies 1\text{-}\textit{bounded entropy conjecture}$$

$$\implies \textit{Peterson-Thom conjecture}$$

Thanks for paying attention!