Strong convergence of tensor products of independent G.U.E. matrices

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Some analytic details

The objects:

- $\mathbb{C}\langle \underline{X}_r \rangle = \mathbb{C}\langle X_1, \ldots, X_r \rangle$, the star-algebra of polynomials in r noncommuting selfadjoint indeterminates;
- $\Psi(z) = \frac{z+i}{z-i}$, Cayley transform sending $\mathbb{C}^-$ onto $\mathbb{D}$, $\mathbb{R}$ onto $\mathbb{T} \setminus \{1\}$;
- $\underline{s}_r = (s_1, \ldots, s_r), \underline{t}_r = (t_1, \ldots, t_r)$ free standard semicirculars, $\underline{u}_r = \Psi(\underline{s}_r), \underline{v}_r = \Psi(\underline{t}_r)$ - Cayley transform applied coordinate-wise;
- $\underline{X}_r^{(N)} = (X_1^{(N)}, \ldots, X_r^{(N)}), \underline{Y}_r^{(N)} = (Y_1^{(N)}, \ldots, Y_r^{(N)})$ normalized independent GUEs, $\underline{U}_r^{(N)} = \Psi(\underline{X}_r^{(N)}), \underline{V}_r^{(N)} = \Psi(\underline{Y}_r^{(N)})$;
- $$S_N = \xi \otimes I_N \otimes I_N + 2\Re \sum_{i=1}^{r} \left(\gamma_i \otimes U_i^{(N)} \otimes I_N + \beta_i \otimes I_N \otimes V_i^{(N)} \right),$$

  $$\mathcal{S} = \xi \otimes 1 \otimes 1 + 2\Re \sum_{i=1}^{r} (\gamma_i \otimes u_i \otimes 1 + \beta_i \otimes 1 \otimes v_i);$$
- $$g_N(z) = \mathbb{E}(\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \mathrm{tr}_N) \left[(zI_m \otimes I_N \otimes I_N - S_N)^{-1} \right],$$

  $$g(z) = (\mathrm{tr}_m \otimes \tau \otimes \tau) \left[(zI_m \otimes 1 \otimes 1 - \mathcal{S})^{-1} \right].$$

Note: sometimes we view $\mathcal{S}$ as an nc function in *variables* $\underline{s}_r$, $\underline{t}_r$.

Expansion in N^{-2}

$$g_N(z) = \mathbb{E}(\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \mathrm{tr}_N)\left[(zI_m \otimes I_N \otimes I_N - S_N)^{-1}\right],$$
$$g(z) = (\mathrm{tr}_m \otimes \tau \otimes \tau)\left[(zI_m \otimes 1 \otimes 1 - \mathcal{S})^{-1}\right].$$

Needed for strong convergence:

$$\left| g_N(z) - g(z) - \frac{E(z)}{N^2} \right| \leq \frac{Q(|\Im z|^{-1})}{N^4}, \quad Q \text{ a polynomial.}$$

Obtained by showing that

$$g_N(z) = g(z) + \frac{E(z)}{N^2} + \frac{\Delta_N(z)}{N^4}, \text{ where}$$

E is the Cauchy transform of a compactly supported real Schwartz distribution whose support is included in $\mathrm{sp}(\mathcal{S})$ (i.e. the domain of analyticity of g is no larger than that of E), and Δ_N is the Cauchy transform of a real compactly supported Schwartz distribution *whose order stays bounded* as $N \to \infty$. (Automatically the two Schwartz distributions "kill" the constants.)

Main analytic tool

Main tool, Félix Parraud (2020): Expansion in powers of N^{-2}:

$$\mathbb{E}\mathrm{tr}_N\left[P\left(\frac{W_N}{\sqrt{N}}\right)\right] = \tau(P(\mathbf{s})) + \sum_{i=1}^{k-1}\frac{\alpha_i(P)}{N^{2i}} + \frac{\alpha_k^N(P)}{N^{2k}}, \tag{1}$$

where $k \in \mathbb{N}, k < N$, $\alpha_1, \ldots, \alpha_{k-1}, \alpha_k^N$ are linear functionals, *which are given by explicit formulas in terms of Voiculescu's free difference quotients* (and more), evaluated in interpolations of free semicirculars (for $\alpha_1, \ldots, \alpha_{k-1}$) or of free semicirculars and GUEs (for α_k^N). Parraud showed α_i, α_i^N are defined on the $*$-algebra $\mathbb{C}\langle \underline{X}_r \rangle = \mathbb{C}\langle X_1, \ldots, X_r \rangle$ of nc polynomials, and on exponentials of nc polynomials. We need it on resolvents.

We use noncommutative functions and Voiculescu's free difference calculus, and the explicit formulas of Parraud for α_1, α_1^N and α_2^N to find E and (show the existence of) Q via Δ_N.

Operations

Voiculescu's free difference quotient (FDQ)

$$\partial_j \colon \mathbb{C}\langle \underline{X}_r \rangle \to \mathbb{C}\langle \underline{X}_r \rangle \otimes \mathbb{C}\langle \underline{X}_r \rangle, \quad \partial_j X_k = \delta_{j,k} 1 \otimes 1,$$

is linear, obeys the Leibniz rule $\partial_j(PQ) = (\partial_j P)[1 \otimes Q] + [P \otimes 1](\partial_j Q)$.

$\mathrm{flip}(A \otimes B) = B \otimes A$.

$\mathrm{ev}_C(A \otimes B) = ACB$.

Fact: If $P \in \mathbb{C}\langle \underline{X}_r \rangle$ and $\underline{x}_r \in \mathcal{A}^r, c \in \mathcal{A}$, then

$$\mathrm{ev}_c((\partial_j P)(\underline{x}_r)) = \Delta_j P(\underline{x}_r, \underline{x}_r)(c) = \lim_{h \to 0} \frac{P(\underline{x}_r + hce_j) - P(\underline{x}_r)}{h}.$$

Here, as usual, $e_j = (0, \ldots, 0, \underbrace{1}_{j^{\text{th}} \text{ position}}, 0, \ldots, 0)$. The above fact extends to many other nc functions, including exponentials and rational functions.

About the alphas

Formula for α_1

$$\alpha_1(P) = \int_0^{+\infty}\int_0^{t_2} \tau(L^{t_1,t_2}(P)(z_t^1, \tilde{z}_t^1, \tilde{z}_t^2, z_t^2))\,\mathrm{d}t_1\mathrm{d}t_2,$$

$$L^{t_1,t_2}(P)(W_1, W_2, W_3, W_4)$$

$$= \frac{1}{2}e^{-t_2-t_1}\sum_{1\le i,j\le r}\Theta^{W_1,W_2,W_3,W_4}(\partial_j \otimes \partial_j(\partial_i(\mathrm{ev}_1 \circ \mathrm{flip} \circ \partial_i P))),$$

$$\Theta^{W_1,W_2,W_3,W_4}\colon A\otimes B\otimes C\otimes D \mapsto B(W_1)A(W_2)D(W_3)C(W_4),$$

$$z_t^1 = (1-e^{-t_1})^{1/2}z^1 + (e^{-t_1}-e^{-t_2})^{1/2}w + e^{-t_2/2}x,$$

$$z_t^2 = (1-e^{-t_1})^{1/2}z^2 + (e^{-t_1}-e^{-t_2})^{1/2}w + e^{-t_2/2}x,$$

$$\tilde{z}_t^1 = (1-e^{-t_1})^{1/2}\tilde{z}^1 + (e^{-t_1}-e^{-t_2})^{1/2}\tilde{w} + e^{-t_2/2}x,$$

$$\tilde{z}_t^2 = (1-e^{-t_1})^{1/2}\tilde{z}^2 + (e^{-t_1}-e^{-t_2})^{1/2}\tilde{w} + e^{-t_2/2}x,$$

where $w, z^1, z^2, \tilde{w}, \tilde{z}^1, \tilde{z}^2, x$ are all free r-semi-circular systems. (For α_1^N, replace x with a tuple $X^{(N)}$ of independent GUE, free from the others.)

Towards extensions

Similar formulas exist for all α_k, α_k^N, $k < N \in \mathbb{N}$, but their complexity increases hugely (we need only α_2 in addition to the previous; the statement of its formula takes 3+ pages).

Important: Formula for α_2 is also a succession of FDQs, flips, tensor-to-free product maps, followed at the very end by an evaluation on r-tuples of selfadjoint variables which are interpolations between various free semicirculars (and a GUE for α_2^N).

Parraud's formulas are defined on nc polynomials P and nc exponentials of polynomials e^Q (note: e^Q needs to be integrable on GUEs — Q must be mostly skew-selfadjoint). In particular, $\alpha_1, \ldots, \alpha_{k-1}, \alpha_k^N$ are all defined on

$$e^{(\pm iX_j+1)y}, 1 \le j \le r, y \in \mathbb{R}.$$

(Exponential always viewed as entire nc function.) Main idea: make sense out of $\alpha_1, \alpha_1^N, \alpha_2, \alpha_2^N$ on

$$\int_{-\infty}^{0} e^{(\pm iX_j+1)y}\,\mathrm{d}y = (\pm iX_j + 1)^{-1}, 1 \le j \le r.$$

Extension I

- Rewrite

$$\mathfrak{U}_j^{\pm 1} := \Psi(X_j)^{\pm 1} = \frac{X_j \pm i}{X_j \mp i} = 1 - 2\int_{-\infty}^{0} e^{(\pm i X_j + 1)y}\,\mathrm{d}y \text{ (nc functions);}$$

- For the specific α_j, use the explicit form in terms of ∂_j, flip, ev, tensor-to-algebraic free product maps, and evaluation on selfadjoints, in order to find a norm on
$$\mathcal{F}(\underline{X}_r) = \langle \mathbb{C}\langle \underline{X}_r \rangle \cup \{e^P : P \in \mathbb{C}\langle \underline{X}_r \rangle \text{ acceptable}\}\rangle$$
in which α_j, α_j^N is continuous and the Riemann integrals above converge;
- Norm-analytic nc functions generally belong to the closure in the norm from the previous item, so Parraud's formula extends to expressions $F_z = (z - P)^{-1}, P = P^*$, and iterations $(z - F_z)^{-1}$, products/sums/compositions of such.

Extension of free difference quotients

Voiculescu's results allow us to apply ∂_j to inverses and compositions. In particular,

$$\partial_j \mathfrak{U}_k^\epsilon = \delta_{j,k}\frac{i}{2}(\mathfrak{U}_k^\epsilon - 1)\otimes(\mathfrak{U}_k^\epsilon - 1), \quad 1 \leq j,k \leq r, \epsilon \in \{\pm 1\}; \tag{2}$$

$$\partial_j (z-P)^{-1} = [(z-P)^{-1}\otimes 1](\partial_j P)[1\otimes(z-P)^{-1}],\ 1\leq j\leq r.$$

Thanks to Parraud,

$$\partial_j e^P = \int_0^1 e^{aP\otimes 1}(\partial_j P)e^{1\otimes(1-a)P}\,\mathrm{d}a.$$

All of the above can be deduced by using power series expansions together with analytic continuation arguments for nc analytic functions. $\mathrm{ev}_C(A\otimes B) = ACB$ and $\mathrm{flip}(A\otimes B) = B\otimes A$ extend the same way.

And repeat the formula...

For an nc function f in the closure wrt the norm "making" α_1 continuous,

$$\alpha_1(f\circ\Psi)=\int_0^{+\infty}\int_0^{t_2}\tau(L^{t_1,t_2}(f)(\Psi(z_t^1),\Psi(\tilde z_t^1),\Psi(\tilde z_t^2),\Psi(z_t^2)))\,\mathrm{d}t_1\mathrm{d}t_2,$$

$$L^{t_1,t_2}(f)(W_1,W_2,W_3,W_4)$$

$$=\frac{1}{2}e^{-t_2-t_1}\sum_{1\le i,j\le r}\Theta^{W_1,W_2,W_3,W_4}\left(\partial_j\otimes\partial_j(\partial_i(\mathrm{ev}_1\circ\mathrm{flip}\circ\partial_i f))\right),$$

$$\Theta^{W_1,W_2,W_3,W_4}: A\otimes B\otimes C\otimes D\mapsto B(W_1)A(W_2)D(W_3)C(W_4)\quad,$$

where

$$z_t^1=(1-e^{-t_1})^{1/2}z^1+(e^{-t_1}-e^{-t_2})^{1/2}w+e^{-t_2/2}x,$$
$$z_t^2=(1-e^{-t_1})^{1/2}z^2+(e^{-t_1}-e^{-t_2})^{1/2}w+e^{-t_2/2}x,$$
$$\tilde z_t^1=(1-e^{-t_1})^{1/2}\tilde z^1+(e^{-t_1}-e^{-t_2})^{1/2}\tilde w+e^{-t_2/2}x,$$
$$\tilde z_t^2=(1-e^{-t_1})^{1/2}\tilde z^2+(e^{-t_1}-e^{-t_2})^{1/2}\tilde w+e^{-t_2/2}x.$$

w, z^1, z^2, $\tilde w$, $\tilde z^1$, $\tilde z^2$, x are free r-semicircular systems. In this writing, ∂_j acts on $f(\underline{\mathfrak{U}}_r,\underline{\mathfrak{U}}_r^*)$ by (2); just a change in notation(!)

Half-way there: we need the alphas on *resolvents of tensor products!* Use Voiculescu's formula for FDQs of inverses and the identification with difference-differential operators.

Alphas on resolvents of tensor products

Closure (simplified/incomplete version)

Need to apply $\mathrm{tr}_m \otimes \alpha_i \otimes \alpha_j$ (or $\mathrm{tr}_m \otimes \alpha_i \otimes \alpha_j^N$), $0 \leq i, j \leq 2$, to $(zI_m \otimes 1 \otimes 1 - \mathcal{S})^{-1}$. Differently said, make sense of $(zI_m \otimes 1 \otimes 1 - \mathcal{S})^{-1}$ *in the closure of* $M_m(\mathbb{C}) \otimes \mathcal{F}(\underline{X}_r) \otimes \mathcal{F}(\underline{X}_r)$ wrt some norm that makes the alphas continuous.

The alphas are compositions of operations $\partial_i, \mathrm{flip}, \mathrm{ev}_{1^s}$, and tensor-to-free products applied to an nc function, followed by evaluations on r-tuples of interpolations of free semicirculars/semicirculars and GUEs, an application of the expectation, and an integration of the interpolating coefficients wrt a finite measure.

Given two successions $\ell_n \circ \cdots \circ \ell_1, \mathfrak{m}_p \circ \cdots \circ \mathfrak{m}_1$ of such operations, define

$$\|f\|_N^{\ell_n \circ \cdots \circ \ell_1 \otimes \mathfrak{m}_p \circ \cdots \circ \mathfrak{m}_1} = \int_{\Omega \times \Omega} \|(\mathrm{id}_{M_m(\mathbb{C})} \otimes \ell_n \circ \cdots \circ \ell_1 \otimes \mathfrak{m}_p \circ \cdots \circ \mathfrak{m}_1)(f)(z_t^1, \ldots, z_t^d)\|_{\min} \mathrm{d}\Pi \otimes \Pi.$$

(Possibly follow with a $\sup_t$.) Here Ω is the classical probability space on which all our GUEs are defined and Π is the probability on it.

Using analyticity of nc functions

Norm-analytic noncommutative functions often belong to closures of the type $\overline{M_m(\mathbb{C}) \otimes \mathcal{F}(\underline{X}_r) \otimes \mathcal{F}(\underline{X}_r)}^{\|\cdot\|_N^{\ell_n \circ \cdots \circ \ell_1 \otimes \mathfrak{m}_p \circ \cdots \circ \mathfrak{m}_1}}$

In the case of $(zI_m \otimes 1 \otimes 1 - \mathcal{S})^{-1}$, we use the following facts:

- $Z \mapsto \int_{-\infty}^{0} e^{(\pm iZ+1)y}\, \mathrm{d}y$ nc function on nc domain $\{-1 < \Im Z < 1\}$;
- $$\lim_{h \to 0} \frac{(z - \mathcal{S}(\Psi(\underline{s}_r + hce_j), \Psi(\underline{t}_r)))^{-1} - (z - \mathcal{S}(\Psi(\underline{s}_r), \Psi(\underline{t}_r)))^{-1}}{h}$$
  $$= \frac{i}{2}(z - \mathcal{S}(\Psi(\underline{s}_r), \Psi(\underline{t}_r)))^{-1}[\gamma_j \otimes (\Psi(s_j) - 1)c(\Psi(s_j) - 1) \otimes 1$$
  $$+ \gamma_j^* \otimes (\Psi(s_j)^{-1} - 1)c(\Psi(s_j)^{-1} - 1) \otimes 1](z - \mathcal{S}(\Psi(\underline{s}_r), \Psi(\underline{t}_r)))^{-1}$$
  $\mathcal{S}(\underline{\mathfrak{U}}_r, \underline{\mathfrak{V}}_r) = \xi \otimes 1 \otimes 1 + \sum_{i=1}^{r}(\gamma_i \otimes \mathfrak{U}_i \otimes 1 + \gamma_i^* \otimes \mathfrak{U}_i^{-1} \otimes 1 + \beta_i \otimes 1 \otimes \mathfrak{V}_i + \beta_i^* \otimes 1 \otimes \mathfrak{V}_i^{-1})$.
  Domains of analyticity don't shrink. Norm changes by $\leq \frac{\mathrm{const}(\mathcal{S})}{|\Im z|}$.
- $z \mapsto (zI_m \otimes 1 \otimes 1 - \mathcal{S})^{-1}$ is analytic on a nbhd of infinity - power series seen before.
- flip is isometric.

E and Δ_N

So, since *in distribution* z_t^j are standard r-tuples of semicirculars, the domain of $E(z)$ is no smaller than the domain of $(zI_m \otimes 1 \otimes 1 - \mathcal{S})^{-1}$ evaluated in tuples of free semicirculars.

And, since finitely many operations are involved in each α, there is a constant C depending on the specific affine polynomial $\mathcal{S}$ and a natural number k such that $|E(z)|, |\Delta_N(z)| < \frac{C}{|\Im z|^k}$. Hence, by a characterization due to Tillmann, E, Δ_N are Cauchy transforms of real compactly supported Schwartz distributions which send constants to zero: $E(z) = \Lambda(x \mapsto (z-x)^{-1}), \Delta_N(z) = \mu_N(x \mapsto (z-x)^{-1})$. Since C and k do not depend on N, but only on the j in α_j^N, the order of μ_N is bounded in N.

Thank you!

$E(z) =$

$$[\mathrm{tr}_m \otimes \tau \otimes \nu_1 + \mathrm{tr}_m \otimes \nu_1 \otimes \tau]((zI_m - \xi) \otimes 1 \otimes 1 - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1 - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I \otimes v_i)^{-1}$$

$$\begin{aligned}
\Delta_N(z) \quad = \quad & \mathbb{E}(\mathrm{tr}_m \otimes \mathrm{tr}_N \otimes \nu_2^{(N)}) \Bigg[\Bigg((zI_m - \xi) \otimes I_N \otimes 1 - \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1 \\
& - \sum_{i=1}^{r_1} \gamma_i^* \otimes U_i^* \otimes 1 - \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes I_N \otimes v_i^* \Bigg)^{-1} \Bigg] \\
& + (\mathrm{tr}_m \otimes \nu_2^{(N)} \otimes \tau) \Bigg[\Bigg((zI_m - \xi) \otimes 1 \otimes 1 - \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1 \\
& - \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1 - \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1 \otimes v_i^* \Bigg)^{-1} \Bigg] \\
& + (\mathrm{tr}_m \otimes \nu_1^{(N)} \otimes \nu_1) \Bigg[\Bigg((zI_m - \xi) \otimes 1 \otimes 1 - \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1 \\
& - \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1 - \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1 \otimes v_i^* \Bigg)^{-1} \Bigg] .
\end{aligned}$$