A phase transition for tails of the free multiplicative convolution powers

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Main message from the first part of the talk

- We were able to characterize behavior of the S-transform of a probability measure on $\mathbb{R}_+$ with regularly varying (right or left) tails.
- Depending on the index $\alpha \geqslant 0$ of the tail function

$$\bar{\mu}(x) := \mu((x, +\infty)) = \frac{L(x)}{x^\alpha},$$

  we observe drastically different behaviors of the S-transform of μ. The four regimes are:
    - $\alpha = 0$,
    - $\alpha \in (0, 1)$,
    - $\alpha = 1$ and $m_1(\mu) = \infty$,
    - $\alpha \geqslant 1$ and $m_1(\mu) < \infty$.
- Proofs rely on classical Tauberian theorems, but their use is not automatic.

Plan

We will apply our results to analyze:

1. tails of free multiplicative convolution powers of a measure with regularly varying tail,
2. tails of $\boxtimes$-infinitely divisible laws when the Lévy measure has regularly varying tail.

In both cases we observe a phase transition between regimes

$$\alpha < 1 \text{ and } \alpha > 1.$$

As a by-product, we obtain the free analogue of Breiman's Lemma:

3. we find tails of $\mu \boxtimes \nu$, when μ has regularly varying tails and ν is "smaller" than μ.

Definitions

- Let $\mathcal{M}_+$ denote the set of Borel probability measures on $\mathbb{R}_+$.
- A measurable function L is *slowly varying* if

$$L(\lambda x) \sim L(x) \text{ for all } \lambda > 0.$$

- By $\boxtimes$ we denote the free multiplicative convolution of probability measures.

$$S_{\mu\boxtimes\nu}(z) = S_\mu(z)S_\nu(z), \qquad z \in (-\varepsilon, 0).$$

- For $\mu \in \mathcal{M}_+$ we define partial semigroup $(\mu^{\boxtimes t})_{t\geqslant 1}$ relative to free multiplicative convolution, where $\mu^{\boxtimes t}$ is the unique measure from $\mathcal{M}_+$ such that

$$S_{\mu^{\boxtimes t}}(z) = S_\mu(z)^t, \qquad z \in (-\varepsilon, 0).$$

Free multiplicative convolution powers

Free multiplicative convolution powers

Theorem ($\alpha = 0$)

Assume that

$$\bar{\mu}(x) \sim \frac{L(\log(x))}{\log(x)^{\beta}}$$

for $\beta > 0$. Then for $t \geqslant 1$,

$$\mu^{\boxtimes t}\big((x, +\infty)\big) \sim t^{\beta}\, \bar{\mu}(x).$$

Free multiplicative convolution powers

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for $\beta > 0$. *Then for* $t \geqslant 1$,

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- Let $\circledast$ denote the classical multiplicative convolution. Then, we have for $n \in \mathbb{N}$,

$$\mu^{\circledast n}((x, +\infty)) \sim n\, \bar{\mu}(x).$$

Free multiplicative convolution powers

Theorem ($\alpha > 0$)

Assume that for $c, \alpha > 0$, we have

$$\bar{\mu}(x) \sim c\, x^{-\alpha}.$$

Then for $t \geqslant 1$,

$$\mu^{\boxtimes t}((x,+\infty)) \sim \begin{cases} c_{t,\alpha}\, x^{-\alpha_t}, & \alpha \in (0,1), \\ c^{t-1} t \log(x)^{t-1}\, \bar{\mu}(x), & \alpha = 1, \\ t\, m_1(\mu)^{\alpha(t-1)}\, \bar{\mu}(x), & \alpha > 1. \end{cases}$$

where

$$\alpha_t = \frac{\alpha}{\alpha + t(1-\alpha)}, \qquad c_{t,\alpha} = \left(c \frac{\pi\alpha}{\sin(\pi\alpha)}\right)^{t/(\alpha+t(1-\alpha))} \frac{\sin(\pi\alpha_t)}{\pi\alpha_t}.$$

Free multiplicative convolution powers

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$$\mu^{\circledast n}((x, +\infty)) \sim \frac{\alpha^{n-1} c^{n-1}}{(n-1)!} \log^{n-1}(x)\, \bar{\mu}(x),$$

Curiosity

Assume that for $c > 0$

$$\bar{\mu}(x) \sim \frac{c}{x^{\alpha}}.$$

Then, for $n \in \mathbb{N}$, $n \geqslant 2$,

$$\lim_{x \to \infty} \frac{\mu^{\boxtimes n}\big((x, +\infty)\big)}{\mu^{\circledast n}\big((x, +\infty)\big)} = \begin{cases} \infty, & \alpha \in (0, 1), \\ n!, & \alpha = 1, \\ 0, & \alpha > 1. \end{cases}$$

Sketch of the proof: $\bar{\mu}(x) \sim c\, x^{-\alpha}$, $\alpha > 1$

$$\mu^{\boxtimes t}\big((x, +\infty)\big) \sim t\, m_1(\mu)^{\alpha(t-1)}\, \bar{\mu}(x).$$

Sketch of the proof: $\bar{\mu}(x) \sim c\, x^{-\alpha}$, $\alpha = 1$

$$\mu^{\boxtimes t}((x, +\infty)) \sim c^t t \log(x)^{t-1}/x =: M(x)/x.$$

$\boxtimes$-infinite divisibility

A measure $\mu \in \mathcal{M}_+$ is said to be $\boxtimes$-infinitely divisible ($\boxtimes$-ID) if, for every $n \in \mathbb{N}$, there exists a measure $\nu_n \in \mathcal{M}_+$ such that $\mu = \nu_n^{\boxtimes n}$.

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Theorem (Bercovici, Voiculescu (1993))

A measure $\mu \in \mathcal{M}_+$ is $\boxtimes$-ID if and only if there exists a finite positive Borel measure σ on the compact space $[0,+\infty]$ and a real number γ such that

$$S_\mu(z) = \exp(v(z)),$$

where v is defined by

$$\begin{aligned} v\left(\frac{z}{1-z}\right) &= \gamma + \int_{[0,+\infty]} \frac{1+tz}{z-t}\sigma(dt) \\ &= \gamma + \sigma(\{0\})\frac{1}{z} - \sigma(\{+\infty\})z + \int_{(0,+\infty)} \frac{1+tz}{z-t}\sigma(dt). \end{aligned}$$

We denote such measure by $\mu_{\boxtimes}^{\gamma,\sigma}$.

Basic properties of $\mu_{\boxtimes}^{\gamma,\sigma}$.

We have

$$m_1(\mu_{\boxtimes}^{\gamma,\sigma}) = \lim_{z\to 0^-} \frac{1}{S_{\mu_{\boxtimes}^{\gamma,\sigma\alpha}}(z)} = \exp(-v(0^-))$$
$$= \exp\big(-\gamma + m_{-1}(\sigma)\big).$$

Lemma

Let $p \in \mathbb{N} \cup \{0\}$. We have

$$m_p(\mu_{\boxtimes}^{\gamma,\sigma}) < +\infty \qquad \textit{if and only if} \qquad m_{-p}(\sigma) < +\infty.$$

Lemma

$$\left(\mu_{\boxtimes}^{\gamma,\sigma}\right)^{-1} = \mu_{\boxtimes}^{-\gamma,\sigma^{-1}}.$$

Corollary

Assume that σ has regularly varying ***left*** *tail with index $-\alpha \leqslant 0$.*
If $\alpha = 1$, assume additionally that limit of $x\sigma\big([0, x^{-1})\big)$ exists as $x \to +\infty$.
Then $\mu_{\boxtimes}^{\gamma,\sigma}$ has regularly varying ***right*** *tail.*

Running example

Let us consider a finite measure on $\mathbb{R}_+$ such that

$$\sigma_\alpha\big([0,x)\big) = c\min\{x^\alpha, d^\alpha\}, \qquad x \geqslant 0, \alpha \geqslant 0.$$

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Theorem

For any $\gamma \in \mathbb{R}$,

$$\mu_{\boxtimes}^{\gamma,\sigma_\alpha}\big((x,\infty)\big) \sim \begin{cases} \left(\frac{c\,\pi\alpha}{\sin(\pi\alpha)}\right)^{1/(1-\alpha)} \frac{1}{\log(x)^{1/(1-\alpha)}} & \textit{for } \alpha \in [0,1), \\ \frac{\pi/(1+c)}{\sin(\pi/(1+c))} d^{c/(1+c)} e^{-\gamma/(1+c)} \frac{1}{x^{1/(1+c)}} & \textit{for } \alpha = 1, \\ e^{-(1+\alpha)\gamma+\alpha d^{\alpha+1}} c \frac{1}{x^\alpha} & \textit{for } \alpha > 1. \end{cases}$$

$\alpha \in [0,1)$

Recall that $S_{\mu_{\boxtimes}^{\gamma,\sigma}}(z) = \exp(v(z))$.

Theorem

Let $\alpha \in [0,1)$. The following two conditions are equivalent:

$$\sigma\big([0,x)\big) \sim c\, x^{\alpha} \qquad \text{as} \qquad x \to 0^+,$$
$$v(-1/x) \sim -\frac{c\,\pi\alpha}{\sin(\pi\alpha)} x^{1-\alpha} \qquad \text{as} \qquad x \to +\infty.$$

Each of these equivalent conditions implies that for all $\gamma \in \mathbb{R}$,

$$\mu_{\boxtimes}^{\gamma,\sigma}\big((x,+\infty)\big) \sim \left(\frac{c\,\pi\alpha}{\sin(\pi\alpha)}\right)^{1/(1-\alpha)} \frac{1}{\log(x)^{1/(1-\alpha)}}.$$

$\alpha = 1$

Theorem

The following two conditions are equivalent:

$$\sigma([0,x)) \sim x\,L(1/x) \qquad \text{as} \qquad x \to 0^+,$$

$$v\left(-\frac{1}{x}\right) = -\int_0^x \frac{L(t)}{t}(1+o(1))dt.$$

Denote $\ell := \lim_{x\to+\infty} L(x) \in [0,+\infty]$. *Then,*

$$x \mapsto \mu_{\boxtimes}^{\gamma,\sigma}((x,+\infty)) \in \mathcal{R}_{-1/(1+\ell)}.$$

$\alpha = 1$, $L \equiv c$

Theorem

(i) *The following two conditions are equivalent:*

$$\sigma\big([0,x)\big) \sim x\,c \qquad \text{as} \qquad x \to 0^+,$$
$$v\left(-\frac{1}{x}\right) = -c\log(x) + \int_0^x o(1)\frac{dt}{t}.$$

- We have
$$S_{\mu_{\boxtimes}^{\gamma,\sigma}}\left(-\frac{1}{x}\right) = e^{v(-1/x)} = \tilde{L}(x)x^{-c},$$
where $\tilde{L}(x) := \exp(\int_1^x o(1)\frac{dt}{t})$ is slowly varying.
- If $\sigma\big([0,x)\big) = c\min\{x,d\}$, then direct calculations show that
$$S_{\mu_{\boxtimes}^{\gamma,\sigma}}(-1/x) = \ldots \sim d^{-c}e^{\gamma}x^{-c}.$$

Theorem

Let $p \in \mathbb{N}$, $\alpha \in (p, p+1)$.
The following two conditions are equivalent:

$$\sigma\big((0,x]\big) \sim x^{\alpha} L(1/x) \qquad as \qquad x \to 0^+,$$
$$v^{(p)}(-1/x) \sim -\Gamma(\alpha+1)\Gamma(p+1-\alpha)x^{p+1-\alpha}L(x).$$

Each of these equivalent conditions implies that

$$\mu_{\boxtimes}^{\gamma,\sigma}\big((x,\infty)\big) \sim m_1(\mu_{\boxtimes}^{\gamma,\sigma})^{\alpha+1}\frac{L(x)}{x^{\alpha}}.$$

Free analogue of Breiman's Lemma

Lemma (Breiman's Lemma)

Let $L \in \mathcal{R}_0$. *If* $\mu, \nu \in \mathcal{M}_+$ *are such that*

$$\mu\big((x, +\infty)\big) \sim x^{-\alpha} L(x) \qquad \textit{and} \qquad m_{\alpha+\varepsilon}(\nu) < +\infty,$$

for $\alpha \geqslant 0$ *and* $\varepsilon > 0$, *then*

$$(\mu \circledast \nu)\big((x, +\infty)\big) \sim m_\alpha(\nu)\bar{\mu}(x).$$

Lemma

Let $L \in \mathcal{R}_0$. *If* $\mu, \nu \in \mathcal{M}_+$ *are such that*

$$\mu\big((x, +\infty)\big) \sim x^{-\alpha} L(x) \qquad \textit{and} \qquad m_{\lfloor \alpha+1 \rfloor}(\nu) < +\infty,$$

for $\alpha \geqslant 0$, *then*

$$(\mu \boxtimes \nu)\big((x, +\infty)\big) \sim m_1^\alpha(\nu)\bar{\mu}(x).$$

Sketch of the proof

The talk was based on the manuscript
B. Kołodziejek, K. Szpojankowski, *A phase transition for tails of the free multiplicative convolution powers*, arXiv:2105.07836

Thank you for your attention.