Berry-Esseen Bounds for Operator-valued Free Limit Theorems

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Joint work with T. Mai

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Outline

- Noncommutative distributions
- Berry-Esseen Bounds: main result
  - Operator-valued free CLT
  - Matrices with operator-valued entries
- Multivariate setting
  - Matrix linear pencils
  - Noncommutative $\mathcal{B}$-valued polynomials

Noncommutative Probability Spaces

An *operator-valued probability space* $(\mathcal{A}, E, \mathcal{B})$ consists of

- a complex algebra $\mathcal{A}$ with unit $1_{\mathcal{A}}$
- a unital complex subalgebra $\mathcal{B}$ of $\mathcal{A}$, which is unitally embedded
- a *conditional expectation* $E : \mathcal{A} \to \mathcal{B}$ a unital $\mathcal{B} - \mathcal{B}$-bimodule map satisfying:
  - $E[b] = b$ for all $b \in \mathcal{B}$
  - $E[b_1 x b_2] = b_1 E[x] b_2$ for all $x \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$.

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Operator-valued C^*- and W^*-probability spaces

$(\mathcal{A}, E, \mathcal{B})$ is called an *operator-valued C^*-probability space* if in addition

- $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras
- $E : \mathcal{A} \to \mathcal{B}$ is positive

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$(\mathcal{A}, \varphi, E, \mathcal{B})$ is called an *operator-valued W^*-probability space* if in addition

- $(\mathcal{A}, \varphi)$ is a tracial W^*-probability space
- $\mathcal{B}$ a von Neumann subalgebra of $\mathcal{A}$

There exists a unique trace preserving $E : \mathcal{A} \to \mathcal{B}$, i.e. $\varphi \circ E = \varphi$.

Framework

- Fix $n \in \mathbb{N}$ and consider two families of self-adjoint elements in $\mathcal{A}$

$$\{x_1, \ldots, x_n\} \quad \text{and} \quad \{y_1, \ldots, y_n\}$$

that are all free with amalgamation over $\mathcal{B}$.

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Question:

How close are the *distributions* of

$$X_n = \sum_{i=1}^{n} x_i \quad \text{and} \quad Y_n = \sum_{i=1}^{n} y_i$$

when $E[x_i] = E[y_i] = 0$ and $E[x_i b x_i] = E[y_i b y_i]$ for all $b \in \mathcal{B}$?

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Denote by $G_x : \mathbb{C}^+ \to \mathbb{C}^-$, $G_x(z) = (z1_{\mathcal{A}} - x)^{-1}$ the resolvent of x.

Cauchy Transforms

For any $z \in \mathbb{C}^+$,

$$\varphi\big(G_x(z)\big) = \int_{\mathbb{R}} \frac{1}{z - t} d\mu_x(t).$$

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- $x_n \to x$ iff $\lim_{n\to\infty} \varphi(G_{x_n}(z)) = \varphi(G_x(z))$ for any $z \in \mathbb{C}^+$.

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The $\mathcal{B}$-valued distribution of x

$$\mu_x^{\mathcal{B}} = \left\{ E[x b_1 x \cdots b_{k-1} x] \;\middle|\; k \in \mathbb{N}_0,\ i_1, \ldots, i_k \in I, b_1, \ldots, b_{k-1} \in \mathcal{B} \right\}.$$

- For any $k \geq 1$, m_k^x is the multilinear map

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$$\lim_{n\to\infty}\left\|(id_k\otimes E)[G_{1_k\otimes x_n}(\mathfrak{b})] - (id_k\otimes E)[G_{1_k\otimes x}(\mathfrak{b})]\right\| = 0.$$

Fix $n \in \mathbb{N}$. The families $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ consist of self-adjoint elements in $\mathcal{A}$ that are all free with amalgamation over $\mathcal{B}$.

$$X_n = \sum_{i=1}^{n} x_i \quad \text{and} \quad Y_n = \sum_{i=1}^{n} y_i$$

whenever $E[x_i] = E[y_i] = 0$ and $E[x_i b x_i] = E[y_i b y_i]$ for all $b \in \mathcal{B}$.

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$$\|E[G_{X_n}(\mathfrak{b})] - E[G_{Y_n}(\mathfrak{b})]\| \leq \ldots$$

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2. Scalar-valued Cauchy transform: for any $\varepsilon > 0$

$$\frac{1}{\pi} \int_{\mathbb{R}} |\operatorname{Im} \varphi(G_{X_n}(t + i\varepsilon)) - \operatorname{Im} \varphi(G_{Y_n}(t + i\varepsilon))| \, \mathrm{d}t \leq \dots$$

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   - Lévy distance: $\mathcal{L}(\mu_{X_n}, \mu_{Y_n}) \leq \ldots$
   - Kolmogorov distance: $\mathrm{Kol}(\mu_{X_n}, \mu_{Y_n}) \leq \ldots$

General Result

- Let $(\mathcal{A}, E, \mathcal{B})$ be an operator-valued C^*-probability space.
- Consider two families $\{x_i\}$ and $\{y_i\}$ of self-adjoint elements in $\mathcal{A}$ that are free with amalgamation over $\mathcal{B}$ and are such that:
  - $E[x_i] = E[y_i] = 0$
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Set $X_n = \sum_{i=1}^n x_i$ and $Y_n = \sum_{i=1}^n y_i$.

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Theorem (B. and Mai '21)

For any $\mathfrak{b} \in \mathbb{H}^+(\mathcal{B})$,

$$\left\| E[G_{X_n}(\mathfrak{b})] - E[G_{Y_n}(\mathfrak{b})] \right\| \leq \| \operatorname{Im}(\mathfrak{b})^{-1} \|^4 A(x) n,$$

where $A(x) = \sqrt{\alpha_2(x)}\left(\sqrt{\alpha_4(x) + \alpha_2(x)^2} + \sqrt{\alpha_4(y) + \alpha_2(x)^2}\right)$ with

$$\alpha_2(x) = \max_{1 \leq i \leq n} \|E[x_i^2]\|, \quad \alpha_4(x) = \max_{1 \leq i \leq n} \sup_{b \in \mathcal{B}, \|b\|=1} \|E[x_i b^* x_i^2 b x_i]\|.$$

General Result

- Let $(\mathcal{A}, \varphi, E, \mathcal{B})$ be an operator-valued W^*-probability space.
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For every $\varepsilon > 0$,

$$\frac{1}{\pi} \int_{\mathbb{R}} |\operatorname{Im} \varphi(G_{X_n}(t + i\varepsilon)) - \operatorname{Im} \varphi(G_{Y_n}(t + i\varepsilon))| \, \mathrm{d}t \leq \frac{A(x)}{\varepsilon^3} n.$$

where $A(x) = \sqrt{\alpha_2(x)}\left(\sqrt{\alpha_4(x) + \alpha_2(x)^2} + \sqrt{\alpha_4(y) + \alpha_2(x)^2}\right)$ with

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Idea of the proof: An operator-valued Lindeberg method

$$E[G_{X_n}(\mathfrak{b})] - E[G_{S_n}(\mathfrak{b})] = \sum_{i=1}^{n} E\big[G_{z_i}(\mathfrak{b}) - G_{z_{i-1}}(\mathfrak{b})\big]$$

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  1. bounds in terms of the fourth moment.

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  1. bounds in terms of the fourth moment.
  2. bounds in terms of the L_2-norm of the Cauchy Transforms to control the integral term and hence to get bounds on the Lévy distance. We use the fact that $-\frac{1}{\pi}\operatorname{Im}\varphi(G_x(t+i\varepsilon))\,\mathrm{d}t$ is a probability measure and

$$\int_{\mathbb{R}} \|G_x(t+i\varepsilon)\|^2_{L^2(\mathcal{A},\varphi)}\,\mathrm{d}t = \frac{\pi}{\varepsilon}.$$

$\mathcal{B}$-Free Central Limit Theorem

- Let $(\mathcal{A}, E, \mathcal{B})$ be an operator-valued C^*-probability space.
- Consider a family $\{x_i\}$ of self-adjoint elements in $\mathcal{A}$ that are free with amalgamation over $\mathcal{B}$ and are such that $E[x_i] = 0$.

Let $X_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i$.

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Operator-Valued Free CLT (Voiculescu '95)

If in addition

- $\sup_{n \in \mathbb{N}} \|m_k^{x_n}\| < \infty$ for all $k \in \mathbb{N}$
- $\exists$ a linear map $\eta : \mathcal{B} \to \mathcal{B}$ such that $\lim_{n\to\infty} n^{-1} \sum_{i=1}^n E[x_i b x_i] = \eta(b)$ for any $b \in \mathcal{B}$

$\mathcal{B}$-Free Central Limit Theorem

- Let $(\mathcal{A}, E, \mathcal{B})$ be an operator-valued C^*-probability space.
- Consider a family $\{x_i\}$ of self-adjoint elements in $\mathcal{A}$ that are free with amalgamation over $\mathcal{B}$ and are such that $E[x_i] = 0$.

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Then

$$X_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j \xrightarrow[n\to\infty]{d} S \quad \text{over } \mathcal{B},$$

where S is an operator-valued semicircular element over $\mathcal{B}$ with variance η.

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For any $n \in \mathbb{N}$, S_n is an *operator-valued semicircular* element with variance given by the completely positive map

$$\eta_n : \mathcal{B} \longrightarrow \mathcal{B}, \qquad \eta_n(b) = \frac{1}{n} \sum_{i=1}^n E[x_i b x_i].$$

Theorem (B. and Mai '21)

For any $\mathfrak{b} \in \mathbb{H}^+(\mathcal{B})$,

$$\big\| E[G_{X_n}(\mathfrak{b})] - E[G_{S_n}(\mathfrak{b})] \big\| \leq \frac{1}{\sqrt{n}} \| \operatorname{Im}(\mathfrak{b})^{-1} \|^4 A(x),$$

where $A(x) = \sqrt{\alpha_2(x)}\big(\sqrt{\alpha_4(x) + \alpha_2(x)^2} + 2\alpha_2(x)\big)$ *with*

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- The bounds don't depend on the operator norm and are merely in terms of the moments of order 2 and 4.

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- **Fully Matricial Extension:**
  Let $1_k \otimes x := \{1_k \otimes x_1, \ldots, 1_k \otimes x_n\} \in (M_k(\mathcal{A}), id_k \otimes E, M_k(\mathcal{B}))$.

$$\left\| (id_k \otimes E)[G_{1_k \otimes X_n}(\mathfrak{b})] - (id_k \otimes E)[G_{1_k \otimes S_n}(\mathfrak{b})] \right\| \leq \frac{k^3}{\sqrt{n}} \| \operatorname{Im}(b)^{-1} \|^4 A(x),$$

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- If we assume same variance, then $S_n = S, \forall n \in \mathbb{N}$. Then for all $k \in \mathbb{N}$, $\|m_k^{X_n} - m_k^S\| \xrightarrow[n\to\infty]{} 0$ and $X_n \xrightarrow[n\to\infty]{d} S$ over $\mathcal{B}$.

$\mathcal{B}$-Free Central Limit Theorem

Theorem (B. and Mai '21)

Consider two operator-valued semicircular elements S_0, S_1 with covariance maps $\eta_0, \eta_1 : \mathcal{B} \to \mathcal{B}$.
Then, for every $k \in \mathbb{N}$ and each $\mathfrak{b} \in \mathbb{H}^+(M_k(\mathcal{B}))$,

$$\|(id_k \otimes E)[G_{1_k \otimes S_1}(\mathfrak{b})] - (id_k \otimes E)[G_{1_k \otimes S_0}(\mathfrak{b})]\| \leq k^2 \| \operatorname{Im}(\mathfrak{b})^{-1} \|^3 \|\eta_1 - \eta_0\|$$

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Under the assumption that

$$\lim_{n\to\infty} \sup_{\|b\|\leq 1} \left\| n^{-1} \sum_{i=1}^{n} E[x_i b x_i] - \eta(b) \right\| = 0$$

we obtain Voiculescu's operator-valued free CLT.

The Lévy and Kolmogorov distances

Let by $\mathcal{F}_\mu : \mathbb{R} \to [0,1],\ t \to \mathcal{F}_\mu(t) = \mu((-\infty, t])$ be the cumulative distribution function of a Borel measure μ.

- The Lévy distance:

  $$\mathcal{L}(\mu,\nu) = \inf\{\varepsilon > 0 \mid \forall t \in \mathbb{R} : \ \mathcal{F}_\mu(t-\varepsilon) - \varepsilon \leq \mathcal{F}_\nu(t) \leq \mathcal{F}_\mu(t+\varepsilon) + \varepsilon\}$$

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- For any $\varepsilon > 0$,

$$\mathcal{L}(\mu,\nu) \leq 2\sqrt{\frac{\varepsilon}{\pi}} + \frac{1}{\pi}\int_{\mathbb{R}} |\operatorname{Im}\varphi(G_\mu(t+i\varepsilon)) - \operatorname{Im}\varphi(G_\nu(t+i\varepsilon))|\,\mathrm{d}t$$

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- If $\mathcal{F}_\nu$ is Hölder continuous with exponent $\beta \in [0,1]$, then

$$L(\mu,\nu) \leq \mathrm{Kol}(\mu,\nu) \leq (C+1)L(\mu,\nu)^\beta.$$

CLT: Kolmogorov distance

- Classical Setting:

$$\mathrm{Kol}(\mu_{X_n}, \mu_{\mathcal{N}}) \leq C \frac{m_3}{\sqrt{n}},$$

where $\mathcal{N}$ is standard Gaussian, C is a constant and m_3 is the absolute third moment of the x_j's.

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- Free Scalar Setting:

$$\mathrm{Kol}(\mu_{X_n}, \mu_s) \leq c\frac{|m_3| + \sqrt{m_4}}{\sqrt{n}}$$

  where s is a semicircular, c is a constant m_3 and m_4 are the third and fourth moments of the x_j's. [Christyakov and Götze '08], [Kargin '07]

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For every $\varepsilon > 0$,

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In particular,

$$\mathcal{L}(\mu_{X_n}, \mu_{S_n}) \leq cA(x)^{1/7} n^{-1/14}.$$

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  By B. and Mai '19, we get a the slightly better bound

  $$\mathrm{Kol}(\mu_{X_n}, \mu_{S_n}) \le Cn^{-\frac{\beta}{2\beta+8}}.$$

Operator-Valued Wigner Matrices

- Consider the $N \times N$ mmatrix

$$A_N = \frac{1}{\sqrt{N}} \begin{pmatrix} \ddots & & * \\ & a_{ii} & \\ a_{ij} & & \ddots \end{pmatrix} \in M_N(\mathcal{A})$$

where $x = \{a_{ij},\, 1 \leq j \leq i \leq N\}$ is family of freely independent elements in $\mathcal{A}$ such that

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Framework

- $(M_N(\mathcal{A}), \mathrm{tr}_N \otimes \varphi, id_N \otimes E, M_N(\mathcal{B}))$

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Operator-Valued Wigner Matrices

- Choose $y = \{c_{ij}, 1 \leq j \leq i \leq N\}$ be a family of circular and semicircular elements that are free with amalgamation over $\mathcal{B}$ s.t.
  - c_{ii} and c_{ij} are semicircular and circular elements over $\mathcal{B}$ respectively
  - $E[c_{ij}] = E[a_{ij}] = 0$
  - $E[c_{ij}^{\varepsilon_1} b c_{ij}^{\varepsilon_2}] = E[a_{ij}^{\varepsilon_1} b a_{ij}^{\varepsilon_2}]$ for all $b \in \mathcal{B}$ and $\varepsilon_1, \varepsilon_2 \in \{1, *\}$

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Set $S_N = \frac{1}{\sqrt{N}} \sum_{1 \leq j \leq i \leq N} \left(e_{ij} \otimes c_{ij} + e_{ji} \otimes c_{ij}^* \right)$

S_N is an operator-valued semicircular element over $\mathcal{D}_N(\mathcal{B}) \subseteq M_N(\mathcal{B})$ with variance map

$$\eta_N : \mathcal{D}_N(\mathcal{B}) \to \mathcal{D}_N(\mathcal{B}), \qquad D \mapsto \eta_N(D),$$

where for any $D = (d_{ij})_{i,j=1}^N \in \mathcal{D}_N^{\mathcal{B}}$,

$$\big(\eta_N(D)\big)_{i,j} = \delta_{i,j} \frac{1}{N} \sum_{r=1}^{i} E[a_{ir} d_{rr} a_{ir}^*] + \delta_{i,j} \frac{1}{N} \sum_{r=i+1}^{N} E[a_{ri}^* d_{rr} a_{ri}].$$

The Multivariate OV Free CLT

Framework

- Let $(\mathcal{A}, \varphi, E, \mathcal{B})$ be an operator valued W^*-probability space.
- Fix $n, d \in \mathbb{N}$. Consider a family x of elements in $\mathcal{A}$ such that

$$(x_1^{(1)}, \ldots, x_1^{(d)}), \ldots, (x_n^{(1)}, \ldots, x_n^{(d)})$$

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Question:
What can be said about the noncommutative joint distribution of

$$X_n = \Big(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j^{(1)}, \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j^{(1)*}, \dots, \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j^{(d)}, \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j^{(d)*}\Big)$$

when $E[x_j^{(k)}] = 0$ and $E[x_j^{(k)} b x_j^{(\ell)}] = 0, \forall b \in \mathcal{B}$?

Noncommutative Distributions

The joint $\mathcal{B}$-valued distribution of $x = (x_i)_{i \in I}$

$$\mu_x^{\mathcal{B}} = \left\{ E[x_{i_1} b_1 x_{i_2} \cdots b_{k-1} x_{i_k}] \;\middle|\; k \in \mathbb{N}_0,\ i_1, \ldots, i_k \in I, b_1, \ldots, b_{k-1} \in \mathcal{B} \right\}.$$

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- For any $k \geq 1$, $m_k^{x_{i_1}, \ldots, x_{i_k}}$ is the multilinear map

$$m_k^{x_{i_1}, \ldots, x_{i_k}} : \mathcal{B}^{k-1} \to \mathcal{B},\ (b_1, \ldots, b_{k-1}) \mapsto E[x_{i_1} b_1 x_{i_2} \cdots x_{i_{k_1}} b_{k-1} x_{i_k}].$$

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- We say that $(x_i^{(n)})_{i \in I} \to (x_i)_{i \in I}$ over $\mathcal{B}$ if for all $k \geq 1$

$$\lim_{n \to \infty} \left\| m_k^{x_{i_1}^{(n)}, \ldots, x_{i_k}^{(n)}} (b_1, \ldots, b_{k-1}) - m_k^{x_{i_1}, \ldots, x_{i_k}} (b_1, \ldots, b_{k-1}) \right\| = 0.$$

The Multivariate OV Free CLT

$$X_n = \Big(\frac{1}{\sqrt{n}}\sum_{j=1}^n x_j^{(1)}, \frac{1}{\sqrt{n}}\sum_{j=1}^n x_j^{(1)*}, \dots, \frac{1}{\sqrt{n}}\sum_{j=1}^n x_j^{(d)}, \frac{1}{\sqrt{n}}\sum_{j=1}^n x_j^{(d)*}\Big)$$

Theorem (Speicher '95)

If we assume the tuples $(x_j^{(1)}, \dots, x_j^{(d)})$'s are *identically distributed* and $E[x_j^{(k)}] = 0$, then

$$X_n \xrightarrow[n\to\infty]{} (C_1, C_1^* \dots, C_d, C_d^*) \qquad \text{over } \mathcal{B},$$

where $\{C_1, \dots, C_d\}$ is a family of centered $\mathcal{B}$-valued circular elements whose covariance $(\eta, \widetilde{\eta})$ is given by the completely positive maps

$$\eta : \mathcal{B} \to M_d(\mathcal{B}),\ b \mapsto [\eta_{k,\ell}(b)]_{k,\ell=1}^d \quad \text{with} \quad \eta_{k,\ell}(b) = E[x_1^{(k)*}\, b\, x_1^{(\ell)}]$$

$$\widetilde{\eta} : \mathcal{B} \to M_d(\mathcal{B}),\ b \mapsto [\widetilde{\eta}_{k,\ell}(b)]_{k,\ell=1}^d \quad \text{with} \quad \widetilde{\eta}_{k,\ell}(b) = E[x_1^{(k)}\, b\, x_1^{(\ell)*}].$$

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$$\mu_x^{\mathcal{B}} = \left\{ E[x_{i_1} b_1 x_{i_2} \cdots b_{k-1} x_{i_k}] \;\middle|\; k \in \mathbb{N}_0,\ i_1, \ldots, i_k \in I, b_1, \ldots, b_{k-1} \in \mathcal{B} \right\}.$$

- For any $k \geq 1$, $m_k^{x_{i_1},\ldots,x_{i_k}}$ is the multilinear map

$$m_k^{x_{i_1},\ldots,x_{i_k}} : \mathcal{B}^{k-1} \to \mathcal{B},\ (b_1, \ldots, b_{k-1}) \mapsto E[x_{i_1} b_1 x_{i_2} \cdots x_{i_{k_1}} b_{k-1} x_{i_k}].$$

- We say that $(x_i^{(n)})_{i\in I} \to (x_i)_{i\in I}$ over $\mathcal{B}$ if for all $k \geq 1$

$$\lim_{n\to\infty} \left\| m_k^{x_{i_1}^{(n)},\ldots,x_{i_k}^{(n)}}(b_1, \ldots, b_{k-1}) - m_k^{x_{i_1},\ldots,x_{i_k}}(b_1, \ldots, b_{k-1}) \right\| = 0.$$

To study the distribution analytically:

joint distribution: $(x_1, \ldots, x_d)$

test function: $f(x_1, \ldots, x_d)$

The Multivariate OV Free CLT

$$X_n = \Big(\frac{1}{\sqrt{n}}\sum_{j=1}^{n} x_j^{(1)}, \frac{1}{\sqrt{n}}\sum_{j=1}^{n} x_j^{(1)*}, \dots, \frac{1}{\sqrt{n}}\sum_{j=1}^{n} x_j^{(d)}, \frac{1}{\sqrt{n}}\sum_{j=1}^{n} x_j^{(d)*}\Big)$$

Test Functions

1. **Matrix Linear Pencils:** For $Q_0 = Q_0^*$, $Q_1, \dots, Q_d \in M_N(\mathbb{C})$

$$g(X_n) = Q_0 \otimes 1_{\mathcal{A}} + \sum_{\ell=1}^{d} (Q_\ell \otimes a_\ell + Q_\ell^* \otimes a_\ell^*).$$

2. **Noncommutative Polynomials:** $p \in \mathcal{B}\langle x_1, x_1^*, \dots, x_d, x_d^* \rangle$ the $\mathbb{C}$-linear span of all $\mathcal{B}$-valued self-adjoint monomials of the form

$$b_0 x_{i_1}^{\varepsilon_1} b_1 x_{i_2}^{\varepsilon_2} b_2 \cdots b_{k-1} x_{i_k}^{\varepsilon_k} b_k \quad k \geq 0, \text{ and } \varepsilon_\ell \in \{1, *\}.$$

Set

$$p(X_n) = p(a_1, a_1^*, \dots, a_d, a_d^*).$$

The Multivariate OV Free CLT: Matrix Linear Pencils

Let $Q_1, \ldots, Q_d \in M_N(\mathbb{C})$,

$$g(X_n) = \sum_{\ell=1}^{d} \left(Q_\ell \otimes \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j^{(\ell)} + Q_\ell^* \otimes \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j^{(\ell)*} \right)$$

$$= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \widetilde{X}_j, \qquad \text{with} \quad \widetilde{X}_j = \sum_{\ell=1}^{d} (Q_\ell \otimes x_j^{(\ell)} + Q_\ell^* \otimes x_j^{(\ell)*}).$$

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Framework

- $(M_N(\mathcal{A}), \mathrm{tr}_N \otimes \varphi, id_N \otimes E, M_N(\mathcal{B}))$

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Framework

- $(M_N(\mathcal{A}), \operatorname{tr}_N \otimes \varphi, id_N \otimes E, M_N(\mathcal{B}))$
- $\widetilde{X}_1, \ldots, \widetilde{X}_n$ are free with amalgamation over $M_N(\mathcal{B})$

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Results from the operator-valued setting **apply!**

The Multivariate OV Free CLT: Matrix Linear Pencils

Choose $\{y_j^{(\ell)}, 1 \leq j \leq n, 1 \leq \ell \leq d\}$ to be a family of $\mathcal{B}$-valued circular elements such that:

- $E[y_j^{(k)}] = 0$
- $E[y_j^{(k)} b y_j^{(\ell)*}] = E[x_j^{(k)} b x_j^{(\ell)*}]$ for any $b \in \mathcal{B}$.
- $(y_1^{(1)}, \ldots, y_1^{(d)}), \ldots, (y_n^{(1)}, \ldots, y_n^{(d)})$ are free over $\mathcal{B}$.

The Multivariate OV Free CLT: Matrix Linear Pencils

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Set $C_j^{(k)} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} y_j^{(k)}$

The Multivariate OV Free CLT: Matrix Linear Pencils

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Set $C_j^{(k)} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} y_j^{(k)}$

$\{C_n^{(1)}, \ldots, C_n^{(d)}\}$ is a family of $\mathcal{B}$-valued centered circular elements with covariance $(\eta_n, \widetilde{\eta}_n)$ given by the completely positive maps

$$\eta_n : \mathcal{B} \to M_d(\mathcal{B}),\ b \mapsto \left[\eta_{k,\ell}^{(n)}(b)\right]_{k,\ell=1}^{d} \quad \text{with} \quad \eta_{k,\ell}^{(n)}(b) = \frac{1}{n} \sum_{j=1}^{n} E[x_j^{(k)*}\, b\, x_j^{(\ell)}]$$

$$\widetilde{\eta}_n : \mathcal{B} \to M_d(\mathcal{B}),\ b \mapsto \left[\widetilde{\eta}_{k,\ell}^{(n)}(b)\right]_{k,\ell=1}^{d} \quad \text{with} \quad \widetilde{\eta}_{k,\ell}^{(n)}(b) = \frac{1}{n} \sum_{j=1}^{n} E[x_j^{(k)}\, b\, x_j^{(\ell)*}]$$

Theorem (B. and Mai '21)

- For any $\mathfrak{b} \in \mathbb{H}^+(M_m(\mathcal{B}))$,

$$\Big\| (\mathrm{id}_m \otimes E)\big[G_{g(X_n)}(\mathfrak{b})\big] - (\mathrm{id}_m \otimes E)\big[G_{g(C_n^{(1)}, C_n^{(1)*}, \ldots, C_n^{(d)}, C_n^{(d)*})}(\mathfrak{b})\big] \Big\| \leq C_g \| \operatorname{Im}(\mathfrak{b})^{-1} \|^4 B(x_n) \frac{1}{\sqrt{n}}.$$

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- Lévy distance:

$$\mathcal{L}\Big(\mu_{g(X_n)}, \mu_{g(C_n^{(1)},C_n^{(1)*},\dots,C_n^{(d)},C_n^{(d)*})}\Big) \leq cB(x_n)^{1/7} n^{-1/14}.$$

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If $(\eta_n, \tilde{\eta}_n) = (\eta, \tilde{\eta})$ is independent of n and $sup_{n\in\mathbb{N}} B(x_n) < \infty$ then

$$\lim_{n\to\infty} \sup_{\substack{\mathfrak{b}\in\mathbb{H}^+(M_m(\mathcal{B}))\\ \operatorname{Im}(\mathfrak{b})\geq\varepsilon 1}} \Big\|(\mathrm{id}_m \otimes E)[G_{g(X_n)}(\mathfrak{b})] - (\mathrm{id}_m \otimes E)[G_{g(C)}(\mathfrak{b})]\Big\| = 0$$

$$(X_n^{(1)}, \dots, X_n^{(d)}) \xrightarrow[n\to\infty]{*-d} (C^{(1)}, \dots, C^{(d)}) \quad \text{over } \mathcal{B}.$$

The Multivariate OV Free CLT: NC Polynomials

Let $p \in \mathcal{B}\langle x_1, x_1^*, \ldots, x_d, x_d^* \rangle$ the $\mathbb{C}$-linear span of all $\mathcal{B}$-valued self-adjoint monomials of the form

$$b_0 x_{i_1}^{\varepsilon_1} b_1 x_{i_2}^{\varepsilon_2} b_2 \cdots b_{k-1} x_{i_k}^{\varepsilon_k} b_k \quad k \geq 0, \text{ and } \varepsilon_\ell \in \{1, *\}.$$

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Consider

$$p(X_n) = p\Big(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j^{(1)}, \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j^{(1)*}, \ldots, \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j^{(d)}, \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j^{(d)*} \Big)$$

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Loss of **linearity**

OV-NC Lindeberg Method doesn't apply

Linearization

- A linear representation of p is a triple $\rho = (u, Q, v)$ with $u^*, v \in \mathbb{C}^m$ and

$$Q := Q(x_1, \ldots, x_d) = Q_0 \otimes 1_{\mathcal{A}} + Q_1 \otimes x_1 + \cdots + Q_d \otimes x_d$$

such that $p = -uQ^{-1}v$.

- We can choose ρ to be self-adjoint. Set

$$\Lambda(\mathfrak{b}) = \begin{bmatrix} \mathfrak{b} & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{Q}_0 = \begin{bmatrix} 0 & v^* \\ v & Q_0 \end{bmatrix} \quad \text{and} \quad \widehat{Q}_j = \begin{bmatrix} 0 & 0 \\ 0 & Q_j \end{bmatrix} \quad \text{for all } j$$

and let

$$\hat{p} = \widehat{Q}_0 \otimes 1_{\mathcal{A}} + \widehat{Q}_1 \otimes x_1 + \widehat{Q}_1^* \otimes x_1^* + \cdots + \widehat{Q}_d \otimes x_d + \widehat{Q}_d^* \otimes x_d^*$$

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- For any $\mathfrak{b} \in \mathbb{H}^+(\mathcal{B})$,

$$G_{p(\cdot)}(\mathfrak{b}) = (\mathfrak{b} - p(\cdot))^{-1} = \left((\Lambda(\mathfrak{b}) - \hat{p}(.))^{-1} \right)_{11} := \left(L_{p(\cdot)}(\mathfrak{b})^{-1} \right)_{11}$$

The Multivariate OV Free CLT: NC Polynomials

Theorem (B. and Mai '21)

Set $M_x = \frac{1}{\sqrt{n}}\|x\| + \sqrt{\alpha_2(x)}$. For any $\mathfrak{b} \in \mathbb{H}^+(\mathcal{B})$,

$$\Big\| E\big[G_{p(X_n)}(\mathfrak{b})\big] - E\big[G_{p(C_n^{(1)},C_n^{(1)*},\dots,C_n^{(d)},C_n^{(d)*})}(\mathfrak{b})\big]\Big\| \leq C_p M_x^{8r_p}\Big(1 + \|\operatorname{Im}(\mathfrak{b})^{-1}\|\Big)^4 B(x_n)\frac{1}{\sqrt{n}}.$$

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1. $\|L_{\hat{p}(x_n)}(\mathfrak{b})\| \leq C M_x^{2r_p}(1 + \|\operatorname{Im}(\mathfrak{b})^{-1}\|)$

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1. $\|L_{\hat{p}(x_n)}(\mathfrak{b})\| \leq C M_x^{2r_p}(1 + \|\operatorname{Im}(\mathfrak{b})^{-1}\|)$
2. For any $\varepsilon > 0$, consider

$$\Lambda_\varepsilon(\mathfrak{b}) = \begin{bmatrix} \mathfrak{b} & & & \\ & i\varepsilon & & \\ & & \ddots & \\ & & & i\varepsilon \end{bmatrix} \in \mathbb{H}^+\big(M_{m+1}(\mathcal{B}\langle x_1, x_1^*, \dots, x_d, x_d^*\rangle)\big)$$

$$E(\mathfrak{b} - p(\cdot))^{-1} = \lim_{\varepsilon \to 0}\big((id_{m+1} \otimes E)(\Lambda(\mathfrak{b}) - \hat{p}(.))^{-1}\big)_{11}$$

The Multivariate OV Free CLT: NC Polynomials

Theorem (B. and Mai '21)

Set $M_x = \frac{1}{\sqrt{n}}\|x\| + \sqrt{\alpha_2(x)}$, then the Lévy distance is bounded

$$\mathcal{L}\Big(\mu_{p(X_n)}, \mu_{p(C_n^{(1)}, C_n^{(1)*}, \ldots, C_n^{(d)}, C_n^{(d)*})}\Big) \leq c M_x^{6r_\rho/7} B(x_n)^{1/7} n^{-1/14}.$$

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History:

- [Speicher '07, Speicher and Mai '13]: multivariate iid CLT, Cauchy transforms and linearizations.
- [Wang '10]: entropic non-microstate iid CLT
- [Fahti and Nelson '17]: multivariate entropic non-microstate ii CLT
- [Jekel and Liu '19]: CLT for $\mathcal{T}$-dependencies

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Projects in progress: extending these results to

- Chebyshev sums
- monotone and boolean independence.

Thank you!