Stochastic integration with respect to the q-Brownian motion

Aurélien Deya (University of Lorraine, France)

Joint work with René Schott

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Objective: develop a robust integration theory with respect to the

q-**Brownian motion**

The q-**Brownian motion** $(X_t^{(q)})_{t\geq 0}$ $(q \in [0,1))$ $\rightarrow$ a natural link between two fundamental objects:

• $(q = 0)$ The **free Brownian motion** $(S_t)_{t\geq 0}$ defined on a **non-commutative probability space** $(\mathcal{A}, \varphi)$.

• $(q \rightarrow 1)$ The **classical Brownian motion** $(W_t)_{t\geq 0}$ defined on a classical probability space $(\Omega, \mathcal{F}, P)$.

Non-commutative probability space? **Free Brownian motion**?

Outline

1. Non-commutative probability spaces
2. q-Brownian motion
3. Integration with respect to the q-Brownian motion
   - First step: Wiener integration $\to$ q-Wiener chaoses
   - Second step: stochastic integration
     - Heuristic considerations $\to$ Pathwise integration theorem
     - Lévy area
     - Some links

Outline

1. **Non-commutative probability spaces**

2. q-Brownian motion

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Non-commutative probability theory

We will call a **non-commutative (NC) probability space** any pair $(\mathcal{A}, \varphi)$ such that:

(i) $\mathcal{A}$ is a unital C^*-algebra:

• $\mathcal{A}$ is a unital algebra over $\mathbb{C}$, equipped with an antilinear $*$-operation $X \mapsto X^*$ satisfying $(X^*)^* = X$ and $(XY)^* = Y^*X^*$ for all $X, Y \in \mathcal{A}$.

• there exists a norm $\|.\| : \mathcal{A} \to [0, \infty[$ which makes $\mathcal{A}$ a Banach space, and such that $\|XY\| \leq \|X\|\|Y\|$ and $\|X^*X\| = \|X\|^2$.

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(ii) φ is a unital positive and faithful trace on $\mathcal{A}$:

• $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional on $\mathcal{A}$

• $\varphi(1) = 1$, $\varphi(X^*X) \geq 0$ for all $X \in \mathcal{A}$

• $\varphi(X^*X) = 0 \Longleftrightarrow X = 0$.

• $\varphi(XY) = \varphi(YX)$ for all $X, Y \in \mathcal{A}$.

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[Nica-Speicher]: *C^*-probability space with faithful trace.*

Non-commutative probability theory

Once endowed with a NC probability space, we call:

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- a **NC process** any path $X : t \mapsto X_t \in \mathcal{A}$.

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Property: Given a *self-adjoint* element $X \in \mathcal{A}$, there exists a unique compactly-supported probability distribution μ_X such that

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We define:

- the **law of a NC random variable** X as the set of its moments

$$\varphi(X^p), \; p \geq 1 \, .$$

- the **law of a NC-process** $(X_t)_{t \geq 0}$ as the set of all of the joint moments

$$\varphi(X_{t_1} X_{t_2} \cdots X_{t_r}), \quad r \geq 1, \; t_1, \ldots, t_r \geq 0 \, .$$

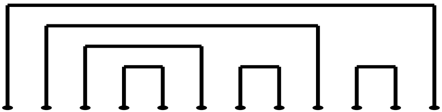
Free Brownian motion $(S_t)_{t\geq 0}$

Definition: A **free Brownian motion** in $(\mathcal{A},\varphi)$ is a NC-process $(S_t)_{t\geq 0}$ in $\mathcal{A}$ whose joint moments are given by the following formula: for all $t_1,\ldots,t_r \geq 0$,

$$\varphi(S_{t_1}\cdots S_{t_r}) = \sum_{\pi\in\mathcal{P}_{\mathbf{0}}(\{1,\ldots,r\})} \prod_{(i,j)\in\pi} \min(t_i,t_j),$$

where $\mathcal{P}_0(\{1,\ldots,r\})$ is the set of the **non-crossing pairings** of $\{1,\ldots,r\}$.

Non-crossing pairing. $\pi := \{(1,12),(2,9),(3,6),(4,5),(7,8),(10,11)\}$.



Outline

q-Brownian motion

A very nice way to interpolate between:

- *the free Brownian motion ($q = 0$)*
- *the classical commutative Brownian motion ($q \to 1$)*

q-Brownian motion

A very nice way to interpolate between:

- *the free Brownian motion* ($q = 0$)
- *the classical commutative Brownian motion* ($q \to 1$)

Recall first that if $(W_t)_{t\geq 0}$ is a classical Brownian motion (on a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$), then its joint moments are given by the **Wick formula**: for all $t_1, \ldots, t_r \geq 0$,

$$\mathbb{E}\left[W_{t_1} \cdots W_{t_r}\right] = \sum_{\pi \in \mathcal{P}(\{1,\ldots,r\})} \prod_{(i,j)\in\pi} \min(t_i, t_j),$$

where $\mathcal{P}(\{1,\ldots,r\})$ is the set of all of the **pairings** of $\{1, \ldots, r\}$.

q-Brownian motion $(X_t^{(q)})_{t\geq 0}$

So far, we have seen that

$$\varphi\big(S_{t_1}\cdots S_{t_r}\big) = \sum_{\pi\in\mathcal{P}_0(\{1,\ldots,r\})}\ \prod_{(i,j)\in\pi} \min(t_i,t_j)\ ,$$

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Let $q\in[0,1)$.

$$\sum_{\pi\in\mathcal{P}(\{1,\ldots,r\})} q^{\mathbf{Cr}(\pi)}\prod_{(i,j)\in\pi}\min(t_i,t_j),$$

where $\mathbf{Cr}(\pi)$ is the number of crossings in π.

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$$\mathbb{E}\big[W_{t_1}\cdots W_{t_r}\big] = \sum_{\pi\in\mathcal{P}(\{1,\ldots,r\})}\prod_{(i,j)\in\pi}\min(t_i,t_j)\ .$$

Definition: Let $q\in[0,1)$. A q-**Brownian motion** in $(\mathcal{A},\varphi)$ is a process $(X_t^{(q)})_{t\geq 0}$ in $\mathcal{A}$ whose joint moments are given by the following formula: for all $t_1,\ldots,t_r\geq 0$,

$$\varphi\big(X_{t_1}^{(q)}\cdots X_{t_r}^{(q)}\big) = \sum_{\pi\in\mathcal{P}(\{1,\ldots,r\})} q^{\mathbf{Cr}(\pi)}\prod_{(i,j)\in\pi}\min(t_i,t_j),$$

where $\mathbf{Cr}(\pi)$ is the number of crossings in π.

Pioneering works on q-Brownian motion (90's):

- M. Bożejko and R. Speicher: *An example of a generalized Brownian motion*. CMP (1991).

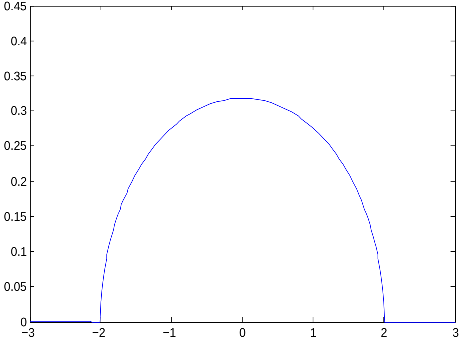
- M. Bożejko and R. Speicher: *Interpolations between bosonic and fermionic relations given by generalized Brownian motions*. Math. Z. (1996).

- M. Bożejko, B. Kümmerer and R. Speicher: *q-Gaussian processes: non-commutative and classical aspects*. CMP (1997).

Remark: The q-Bm can also be defined for $-1 < q < 0$, but we will not consider this case.

Definition: The law of $X_1^{(q)}$ is called the **q-Gaussian distribution**. It admits a density.

$$q = 0$$



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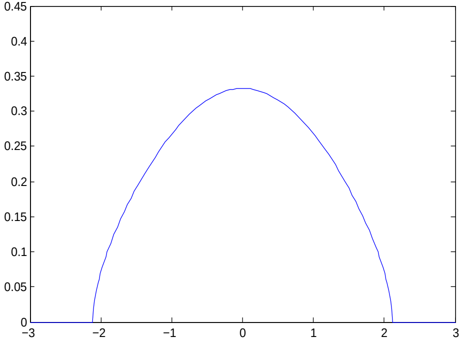
$$q = 0.1$$



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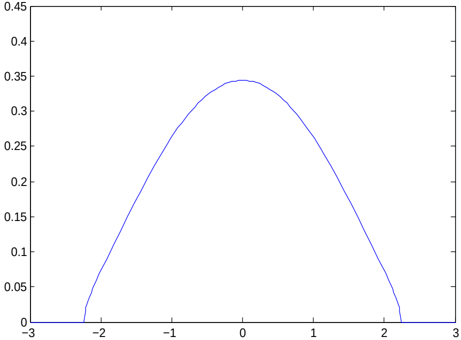
$$q = 0.2$$



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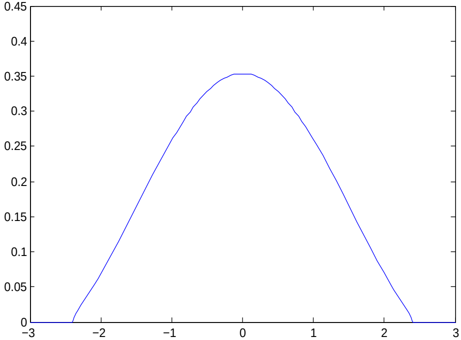
$$q = 0.3$$



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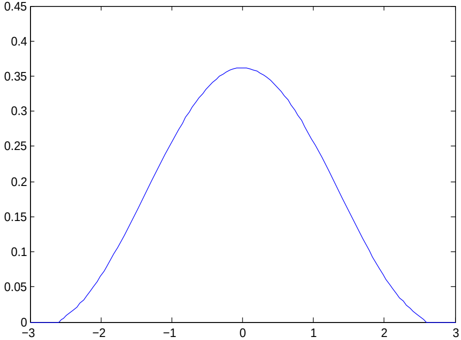
$$q = 0.4$$



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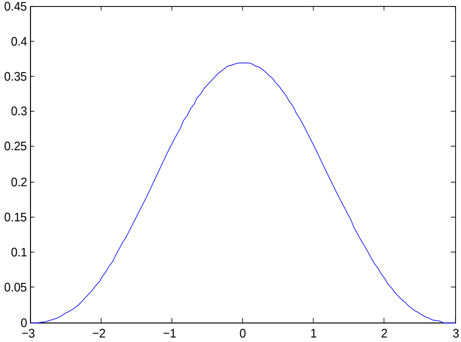

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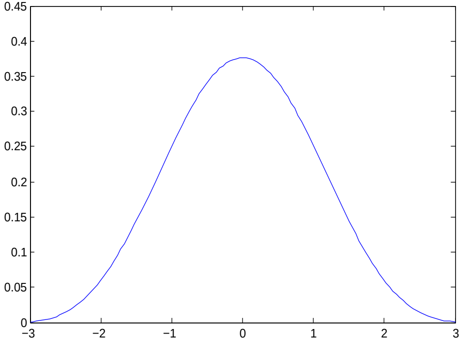
$$q = 0.6$$



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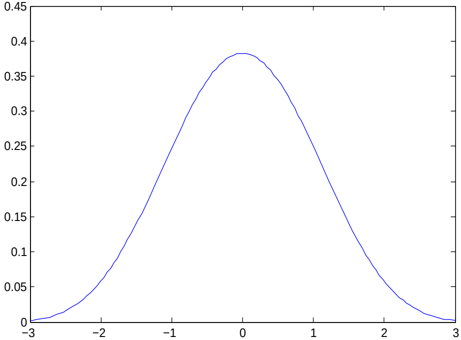

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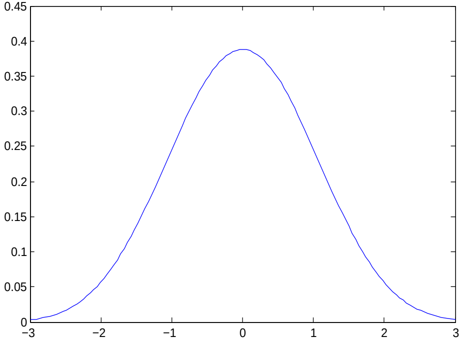

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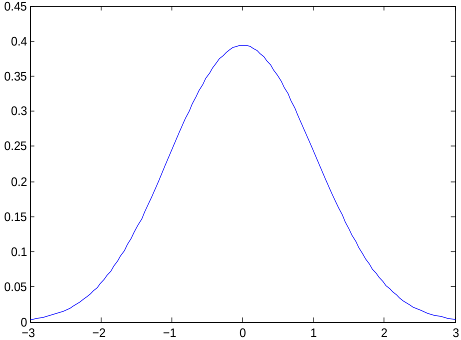

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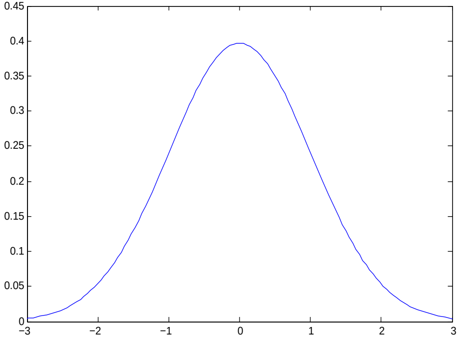
$$q \approx 1$$



q-Brownian as limit of random matrices

($q = 0$) Consider two independent families $(x(i,j))_{i \geq j}$ and $(\tilde{x}(i,j))_{i \geq j}$ of independent Brownian motions, and let $M^{(d)}$ be the process with values in the space of the $(d \times d)$-Hermitian matrices and with entries

$$M_t^{(d)}(i,j) := \frac{1}{\sqrt{2d}}\big(x_t(i,j) + \imath\, \tilde{x}_t(i,j)\big) \quad \text{for } 1 \leq j < i \leq d \ ,$$

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Theorem (Voiculescu, 91): for all times $t_1, \ldots, t_r \geq 0$,

$$\frac{1}{d}\mathbb{E}\Big[\mathrm{Tr}\big(M_{t_1}^{(d)} \cdots M_{t_r}^{(d)}\big)\Big] \xrightarrow{d \to \infty} \varphi\big(X_{t_1} \cdots X_{t_r}\big) \ ,$$

where $X = X^{(0)}$ is a free Brownian motion.

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There exist "q-extensions" of the above result:

• P. Sniady: Gaussian random matrix models for q-deformed Gaussian random variables. *CMP* (2001).

• M. Pluma and R. Speicher: A dynamical version of the SYK Model and the q-Brownian Motion. *Preprint* (2020).

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Objective

Let $0 \leq q < 1$ and $X^{(q)}$ be a q-Brownian motion $\longrightarrow$ Define the integral

$$\int H_s \,\mathrm{d}X_s^{(q)}\, K_s$$

for processes $H, K : \mathbb{R}_+ \to \mathcal{A}$ in a class to be determined.

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- define and study stochastic differential equations

$$dY_t = f(Y_t)\,\mathrm{d}X_t^{(q)}\, g(Y_t)$$

$\Rightarrow$ rigorous interpretation of equations driven by growing random matrices
$\Rightarrow$ define interesting objects as solutions to stochastic differential equations

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- Mathematical challenge: for each $0 \le q < 1$, it holds that

$$\|X_t^{(q)} - X_s^{(q)}\| = c_q \, |t - s|^{\frac{1}{2}} \quad .$$

$\Rightarrow$ the path $t \mapsto X_t^{(q)}$ is not differentiable

Outline

Chaoses of $X^{(q)}$

Fix $0 \leq q < 1$ and $n \in \mathbb{N}$. For every $f \in L^2(\mathbb{R}_+^n)$, we can define the **multiple integral**

$$\boxed{I_n^{X^{(q)}}(f) = \int_{\mathbb{R}_+^n} f(t_1, \ldots, t_n)\, \mathrm{d}X_{t_1}^{(q)} \cdots \mathrm{d}X_{t_n}^{(q)}}$$

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(2) Extend linearly to simple functions.

(3) Use a (non-trivial!) isometry property

$$\varphi\big(I_n^{X^{(q)}}(f) I_m^{X^{(q)}}(g)^*\big) = \delta_{n,m}\langle f, g\rangle_q \quad (\langle .,. \rangle_q \text{ **scalar product**!})$$

to extend the integral (in $L^2(\varphi)$) to integrands $f \in L^2(\mathbb{R}_+^n), g \in L^2(\mathbb{R}_+^m)$.

Multiplication formula

Proposition (D.-Schott). Let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be *fully-symmetric* functions. Then it holds that

$$I_n^{X^{(q)}}(f) I_m^{X^{(q)}}(g) = \sum_{k=0}^{n \wedge m} [k]_q! \binom{n}{k}_q \binom{m}{k}_q I_{n+m-2k}^{X^{(q)}}(f \otimes_k g) \ ,$$

where

$$(f \otimes_k g)(t_1, \dots, t_{n+m-2k}) :=$$
$$\int_{\mathbb{R}_+^k} f(t_1, \dots, t_{n-k}, s_1, \dots, s_k) g(s_1, \dots, s_k, t_{n-k+1}, \dots, t_{n+m-2k}) \, ds_1 \cdots ds_k$$

and where we have used the q-binomial coefficients

$$[n]_q := 1 + q + \cdots + q^{n-1} \quad , \quad [n]_q! := [n]_q [n-1]_q \cdots [1]_q \ ,$$

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! \, [n-k]_q!} \ .$$

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Pioneering works

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Free Brownian motion ($q = 0$)

• P. Biane and R. Speicher: *Stochastic calculus with respect to free Brownian motion and analysis on Wigner space*. PTRF (1998).

• P. Biane and R. Speicher: *Free diffusions, free entropy and free Fisher information*. AIHP (2001).

$\Longrightarrow$ use the free independence properties of $X^{(0)}$

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q-Brownian motion

• C. Donati-Martin: *Stochastic integration with respect to q-Brownian motion*. PTRF (2003).

$\Longrightarrow$ $L^2(\varphi)$-construction (the integral may belong to a completion of $\mathcal{A}$)

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Let P be a polynomial, and denote by X the q-Brownian motion.

Objective: using ideas from rough paths theory, try to define the integral

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$$\int_s^t Y_u\,\mathrm{d}Z_u = \lim_{|\Delta_{st}|\to 0} \sum_{(t_i)\in\Delta_{st}} Y_{t_i}(Z_{t_{i+1}} - Z_{t_i}) \quad \text{in } \mathcal{A}.$$

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Idea: corrected Riemann sums

$$\int_s^t P(X_u)\mathrm{d}X_u := \lim_{|\Delta_{st}|\to 0} \sum_{(t_i)\in\Delta_{st}} \left\{P(X_{t_i})(X_{t_{i+1}} - X_{t_i}) + C_{t_i,t_{i+1}}\right\},$$

with some natural "second-order" correction term C.

Step 1: identify C (heuristic)

Formal Taylor expansion:

$$\int_s^t P(X_u)\mathrm{d}X_u$$

$$= P(X_s)(X_t - X_s) + \int_s^t \nabla P(X_s)(X_u - X_s)\mathrm{d}X_u + \int_s^t Y_{s,u}\mathrm{d}X_u\,.$$

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$$\Big\| \int_s^t Y_{s,u}\mathrm{d}X_u \Big\| \leq c\,|t-s|^{\frac{3}{2}}\ ,$$

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$\Longrightarrow$ **A natural (potential!) definition:**

$$\int P(X_t)\mathrm{d}X_t \text{ ``:=''}$$

$$\lim_{|\Delta_{st}|\to 0} \sum_{(t_i)\in\Delta_{st}} \Big\{P(X_{t_i})(X_{t_{i+1}} - X_{t_i}) + \int_{t_i}^{t_{i+1}} \nabla P(X_{t_i})(X_u - X_{t_i})\mathrm{d}X_u\Big\}.$$

Step 2: understand C (heuristic)

For instance, when $P(x) = x^p$,

$$C_{s,t} = \int_s^t \nabla P(X_s)(X_u - X_s)\mathrm{d}X_u$$

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$\Longrightarrow$ to define the correction C, one must be able to interpret the *iterated integral* (or *Lévy area*)

$$\mathbb{X}^2_{s,t}[A] := \int_s^t \int_s^u \mathrm{d}X_v \, A \, \mathrm{d}X_u \qquad \text{for every } A \in \mathcal{A}_s \, ,$$

where $\mathcal{A}_s = \mathcal{A}_s^X$ is the unital subalgebra generated by $\{X_r\}_{0 \leq r \leq s}$.

Rough paths integration theorem (loose form). (Gubinelli, D.-Schott)

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Then, for every polynomial P, we can rigourously define, in $\mathcal{A}$,

$$\int_s^t P(X_u)\mathrm{d}X_u := \lim \sum_{(t_i)\in\Delta_{[s,t]}} \Big\{ P(X_{t_i})(X_{t_{i+1}} - X_{t_i}) + \int_{t_i}^{t_{i+1}} \nabla P(X_{t_i})(X_u - X_{t_i})\mathrm{d}X_u \Big\}.$$

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Besides:

(i) We can extend this result to define the more general integral

$$\int_s^t H_u \, \mathrm{d}X_u \, K_u$$

for $H, K : [0, T] \to \mathcal{A}$ in a larger class (the so-called *controlled paths*).

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$$dY_t = f(Y_t) \, \mathrm{d}X_t \, g(Y_t).$$

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(iii) Continuity: $Y = \Phi(X, \mathbb{X}^2)$, with Φ continuous.

Outline

Construction of the Lévy area

Question: How to define (for every fixed $A \in \mathcal{A}_s$)

$$\mathbb{X}^2_{s,t}[A] := \int_s^t \int_s^u \mathrm{d}X_v \, A \, \mathrm{d}X_u \qquad ?$$

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Regularization procedure: consider the linear interpolation of X, that is

$$X^{(n)}_t := X_{t^n_i} + n(t - t^n_i)\{X_{t^n_{i+1}} - X_{t^n_i}\} \quad \text{for } t \in [t^n_i, t^n_{i+1}], \; t^n_i = \frac{i}{n}.$$

Then we set, for all $0 \leq s \leq t$ and $A \in \mathcal{A}$,

$$\mathbb{X}^{2,(n)}_{st}[A] := \int_s^t \int_s^u \mathrm{d}X^{(n)}_v \, A \, \mathrm{d}X^{(n)}_u \; ,$$

where the integral is understood **in the classical Lebesgue sense**.

Objective: Study the convergence of $\mathbb{X}^{2,(n)}_{st}[A]$ as $n \to \infty$.

Construction of the Lévy area

Proposition (D.-Schott): For all $0 \leq s \leq t$ and $A \in \mathcal{A}_s$,

the sequence $\mathbb{X}^{2,(n)}_{st}[A]$ converges in $(\mathcal{A}, \|.\|)$ as $n \to \infty$.

The limit provides us with a natural interpretation of

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Proof. We show that $\big(\mathbb{X}^{2,(n)}_{st}[A]\big)_{n\geq 1}$ is a Cauchy sequence in $(\mathcal{A}, \|.\|)$, using the identity

$$\|U\| = \lim_{p\to\infty} \varphi\big((UU^*)^p\big)^{\frac{1}{2p}} \, .$$

Remark: For the moment, we fail to extend the proof to $q \in (-1, 0)$.

Outline

1. Non-commutative probability spaces

2. q-Brownian motion

3. **Integration with respect to the q-Brownian motion**
   - First step: Wiener integration $\rightarrow$ q-Wiener chaoses
   - Second step: stochastic integration
     - Heuristic considerations $\rightarrow$ Pathwise integration theorem
     - Lévy area
     - Some links

Stratonovich interpretation

Proposition. Consider the approximation

$$X_t^{(n)} := X_{t_i^n} + n(t - t_i^n)\{X_{t_{i+1}^n} - X_{t_i^n}\} \quad \text{for } t \in [t_i^n, t_{i+1}^n],\ t_i^n = \frac{i}{n}.$$

Then one has

$$\int_0^1 P(X_u^{(n)})(\mathrm{d}X_u^{(n)})Q(X_u^{(n)}) \overset{n\to\infty}{\to} \int_0^1 P(X_u)\mathrm{d}X_u Q(X_u) \quad \text{in } \mathcal{A}.$$

$\Rightarrow$ We rather denote the integral as $\int_0^1 P(X_u)(\circ\mathrm{d}X_u)Q(X_u)$.

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Wong-Zakai theorem. Let $Y^{(n)}$ be the classical (Lebesgue) solution of

$$dY_t^{(n)} = f(Y_t^{(n)})\,\mathrm{d}X_t^{(n)}\,g(Y_t^{(n)}),$$

where f, g are regular functions.

Then $Y^{(n)} \stackrel{n\to\infty}{\longrightarrow} Y$ in $\mathcal{A}$, where Y is the ("Stratonovich") solution of

$$dY_t = f(Y_t)\,(\circ\mathrm{d}X_t)\,g(Y_t).$$

Stratonovich interpretation

Itô-Stratonovich formula. For every regular f, one has, in $L^2(\varphi)$,

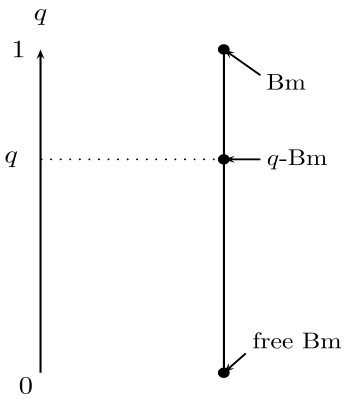
$$\begin{aligned} f(X_t) - f(X_s) &= \int_s^t \partial f(X_u)\sharp(\circ \mathrm{d}X_u) \\ &= \int_s^t \partial f(X_u)\sharp \mathrm{d}X_u + \frac{1}{2}\int_s^t [\mathrm{Id} \times \Gamma_q \times \mathrm{Id}](\partial^2 f(X_u))\,\mathrm{d}u \end{aligned}$$

where:

- $\int_s^t \partial f(X_u)\sharp(\circ \mathrm{d}X_u)$ is the Stratonovich integral
- $\int_s^t \partial f(X_u)\sharp \mathrm{d}X_u$ is the Itô integral (cf Donati-Martin's work)
- $(\Gamma_q)_{q\in[0,1)}$ is a family of operators that interpolates between

$$\Gamma_0(U) = \varphi(U) \quad \text{and} \quad \text{“}\Gamma_1(U) = U\text{”}.$$

Another (future) possible extension: the fractional q-Brownian (or (H, q)-Bm)



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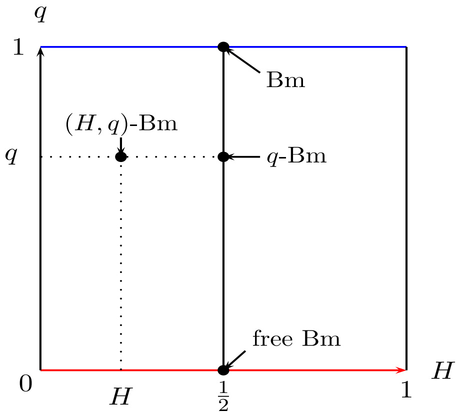

Matrix approximation ($q = 0$)

We focus on the free case: $X := X^{(0)}$.

For $d \geq 1$, let $M^{(d)}$ be a d-dimensional Hermitian Brownian motion.

Proposition (D.): For all polynomials P, Q, one has

$$\int_0^1 P(M_u^{(d)})(\circ \mathrm{d} M_u^{(d)}) Q(M_u^{(d)}) \xrightarrow{d \to \infty} \int_0^1 P(X_u)(\circ \mathrm{d} X_u) Q(X_u)$$

in the sense of non-commutative probability.