Cylic (Monotone and Boolean) Independence

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Reminder: Monotone Independence

We consider a non commutative probability space $(\mathcal{A}, \phi)$. $\mathcal{A}$ is an algebra ϕ is a linear functional such that $\phi(1_{\mathcal{A}}) = 1$.

Definition

For two subalgebras $(\mathcal{A}_i)_{i\in\{1,2\}}$ of $\mathcal{A}$. The algebra A_1 is said to be **monotone** independent of the algebra A_2 if for all $a_1, a_3, .., a_{2n-1} \in A_1, a_0, a_2, ..., a_{2n} \in A_2$, we have

$$\phi(a_0 a_1 a_2 \cdots a_{2n}) = \phi(a_1 a_3 \cdots a_{2n-1})\phi(a_0)\phi(a_2)\phi(a_4)\cdots\phi(a_{2n}).$$

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Remarks:

- We may assume $1_A \in A_2$. However, A_1 is in general assumed to be non-unital.
- ϕ is assumed to be non-tracial.

Monotone Independence

We consider the pair $(\mathcal{M}_n, \phi_1)$, where

$$\mathcal{M}_n := \{\text{Matrices of size } n \times n\}$$

and, for a matrix $M \in \mathcal{M}_n$, ϕ_1 denotes

$$\phi_1(M) = M_{11}.$$

Example

In $(\mathcal{M}_{nm}, \phi_1)$, the matrices $P_n \otimes A$ and $B \otimes P_m$ are Boolean independent with respect ϕ_1.

Example

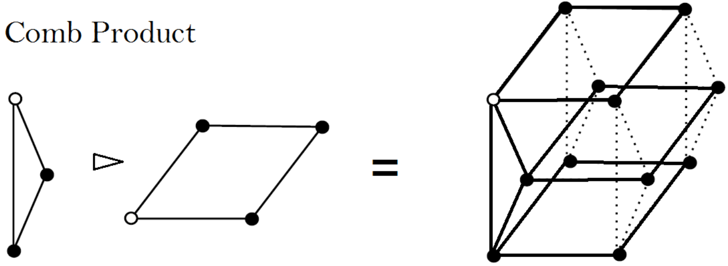
In $(\mathcal{M}_{nm}, \phi_1)$, the matrices $I_n \otimes A$ and $B \otimes P_m$ are monotone independent with respect ϕ_1.

Comb Product of Rooted Graph products

Example

In $(\mathcal{M}_{nm}, \phi_1)$, the matrices $I_n \otimes A$ and $B \otimes P_m$ are monotone independent with respect ϕ_1.

When A and B are adjacency matrices of graphs G_1 and G_2, $I_n \otimes A + B \otimes P_m$ corresponds to the adjacency matrix of the comb product of G_1 and G_2:



Monotone convolution

We denote the Cauchy transform and its reciprocal.

$$G_a(z) = \sum_{n=0}^{\infty} z^{-n-1}\varphi(z^n), \qquad F_a(z) = 1/G_a(z)$$

Let $a, b \in (\mathcal{A}, \varphi)$. If a is monotone independent from b, then

$$F_{(a+b)} = F_b(F_a(z)).$$

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To see this, expand $(a+b)^n$ as follows

$$(a+b)^n = \; = \sum_{k=0}^{n} \sum_{\substack{q_0 \ldots, q_k \geq 0 \\ q_0+\cdots+q_k=n-k}} b^{q_0} a b^{q_1} \cdots b^{q_k} a b^{q_k}.$$

and appy φ, to obtain

$$\varphi((a+b)^n) \;=\; \sum_{k=0}^{n} \sum_{\substack{q_0 \ldots, q_k \geq 0 \\ q_0+\cdots+q_k=n-k}} \varphi(a^k)\varphi(b^{q_0})\varphi(b^{q_1})\cdots\varphi(b^{q_k}).$$

Multipliying by z^{-n-1} and summing over $n \geq 0$ we obtain

$$\begin{aligned}
G_{a+b}(z) &= \sum_{n=0}^{\infty} z^{-n-1}\varphi((a+b)^n) \\
&= \sum_{n=0}^{\infty} z^{-n-1} \sum_{k=0}^{n} \sum_{\substack{q_0 \dots, q_k \geq 0 \\ q_0+\cdots+q_k=n-k}} \varphi(a^k)\varphi(b^{q_0})\varphi(b^{q_1})\cdots\varphi(b^{q_k}) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \sum_{\substack{q_0, \dots, q_k \geq 0 \\ q_0+\cdots+q_k=n-k}} \varphi(a^k)\frac{\varphi(b^{q_0})}{z^{q_0+1}}\frac{\varphi(b^{q_1})}{z^{q_1+1}}\cdots\frac{\varphi(b^{q_k})}{z^{q_k+1}} \\
&= \sum_{k=0}^{\infty} \sum_{q_0, \dots, q_k \geq 0} \varphi(a^k)\frac{\varphi(b^{q_0})}{z^{q_0+1}}\frac{\varphi(b^{q_1})}{z^{q_1+1}}\cdots\frac{\varphi(b^{q_k})}{z^{q_k+1}} \\
&= \sum_{k=0}^{\infty} \varphi(a^k) G_b(z)^{k+1} = \sum_{k=0}^{\infty}\varphi(a^k)\Big(\frac{1}{F_b(z)}\Big)^{k+1} \\
&= G_a(F_b(z)).
\end{aligned}$$

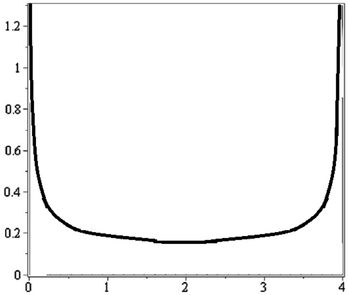
Taking reciprocals, we obtain the desired result.

Monotone CLTs

Theorem (Muraki 2001)

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of **monotone** *independent random random variables, i.d with $\phi(x_n) = 0$ and $\phi(x_n^2) = 1$. Then*

$$\frac{x_1 + \ldots + x_N}{\sqrt{N}} \to A.$$



$$\frac{1}{\pi\sqrt{4-t^2}} \quad |t| < 2.$$

Cyclic Monotone Independence
(Collins,Hasebe, Sakuma (2018))

Definition

Let $(\mathcal{C}, \tau)$ be a non-commutative probability space with a tracial weight ω. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$ be $*$-subalgebras such that $1_{\mathcal{C}} \in \mathcal{B}$. We say that the pair $(\mathcal{A}, \mathcal{B})$ is **cyclically monotonically independent** with respect to (ω, τ) if for any $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathcal{A}$, $b_1, \ldots, b_n \in \mathcal{B}$, we have

$$\omega(a_1 b_1 a_2 b_2 \cdots a_n b_n) = \omega(a_1 a_2 \cdots a_n)\tau(b_1)\tau(b_2)\cdots\tau(b_n).$$

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$$\omega(a_1 b_1 a_2 b_2 \cdots a_n b_n) = \omega(a_1 a_2 \cdots a_n)\tau(b_1)\tau(b_2)\cdots\tau(b_n).$$

Remark: traciality of ω implies

$$\omega(b_0 a_1 b_1 a_2 b_2 \cdots a_n b_n) = \omega(a_1 a_2 \cdots a_n)\tau(b_1)\tau(b_2)\cdots\tau(b_n b_0).$$

In monotone independence the analog monomial would be evaluatued as

$$\omega(b_0 a_1 b_1 a_2 b_2 \cdots a_n b_n) = \omega(a_1 a_2 \cdots a_n)\tau(b_0)\tau(b_1)\tau(b_2)\cdots\tau(b_n).$$

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$$\omega(a_1 b_1 a_2 b_2 \cdots a_n b_n) = \omega(a_1 a_2 \cdots a_n)\tau(b_1)\tau(b_2) \cdots \tau(b_n).$$

Remark:

- **Note that in this framework nothing is said about the joint moments w.r.t** τ**, eg.** $\tau(a_1 b_1)$.
- **Also** $\omega(a_1 b_1 a_2 b_2 \cdots a_n b_n)$ **does not involve** $\tau(a_i^l)$

For applications to Random Matrices, CHS use a slightly more flexible framework:

1) The tracial weight is ω defined in a $*$-subalgebra $\mathrm{D}(\omega) \subset \mathcal{C}$.

2) The $*$-ideal

$$\mathrm{I}_{\mathcal{B}}(\mathcal{A}) := \mathrm{span}\,\{b_0 a_1 b_1 \cdots a_n b_n \;:\; n \in \mathbb{N},\, a_1, \ldots, a_n \in \mathcal{A}, b_0, \ldots, b_n \in \mathcal{B}\}$$

is assumed to be in $\mathrm{D}(\omega)$.

Application of Random Matrices

Theorem (CHS 2018)

Let $n \in \mathbb{N}$. Let $U = U(n)$ be an $n \times n$ Haar unitary random matrix and $A_i = A_i(n)$, $B_j = B_j(n)$, $i = 1, \ldots, k$, $j = 1, \ldots, \ell$ be $n \times n$ random matrices such that

1. $((A_1, \ldots, A_k), \mathbb{E} \otimes \mathrm{Tr}_n)$ *converges in distribution to a k-tuple of trace class operators as $n \to \infty$.*
2. $((B_1, \ldots, B_\ell), \mathbb{E} \otimes tr_n)$ *converges in distribution to an ℓ-tuple of elements in a non-commutative probability space as $n \to \infty$.*
3. $\{A_1, \ldots, A_k\}, \{B_1, \ldots, B_\ell\}, U$ *are independent.*

Then the pair $(\{A_1, \ldots, A_k\}, \{UB_1U^*, \ldots, UB_\ell U^*\})$ is **asymptotically cyclically monotone** *with respect to $(\mathbb{E} \otimes \mathrm{Tr}_n, \mathbb{E} \otimes tr_n)$.*

Theorem (CHS 2008)

Let $(\mathcal{A}, \omega, \tau)$ be a non-commutative probability space with tracial weight ω. Suppose the pair (a, b) is cyclically monotone with respect to (ω, τ). Then

1. *If a, b are selfadjoint, $p = \sqrt{\tau(b^2)} + \tau(b)$ and $q = -\sqrt{\tau(b^2)} + \tau(b)$ then*

$$\mathrm{EV}(ab + ba) = (p\,\mathrm{EV}(a)) \sqcup (q\,\mathrm{EV}(a)).$$

2. *If a, b are selfadjoint and $r = \sqrt{\tau(b^2) - \tau(b)^2}$, then*

$$\mathrm{EV}(\mathrm{i}(ab - ba)) = (r\,\mathrm{EV}(a)) \sqcup (-r\,\mathrm{EV}(a)).$$

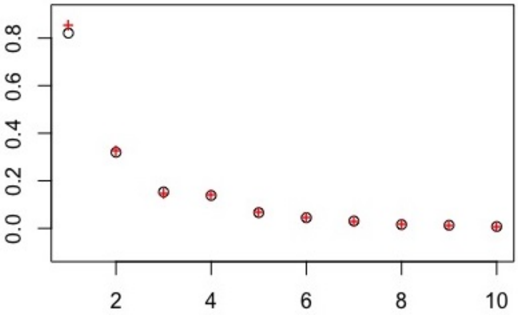
Some comments:

- The proof of the above theorem is based on combinatorial arguments.
- In joint work with Adrian Celestino [1] we give a general procedure to find the spectrum of any polynomial in $I_{\mathcal{B}}(A)$.
- If a polynomial does not belong to $I_{\mathcal{B}}(A)$. things are much harder. In particular $a+b$ needs some work.

[1] Arizmendi, O., & Celestino, A. Polynomial with cyclic monotone elements with applications to Random Matrices with discrete spectrum. Random Matrices: Theory and Applications, 10(02), 2150020.

Example

Take $D = \text{diag}(2^{-1}, 2^{-2}, \ldots, 2^{-n})$, G a GUE $A = UDU^*$, $B = G^2$.
The spectrum $X = A + BABAB$ may be predicted:

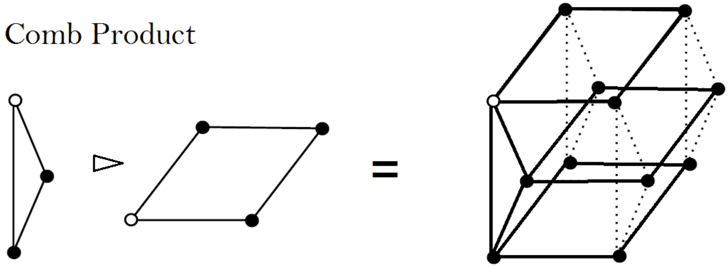


10 eigenvalues of X (black circle) and of $a + babab$ (red triangle)

Cyclic monotone independence and comb product

Example

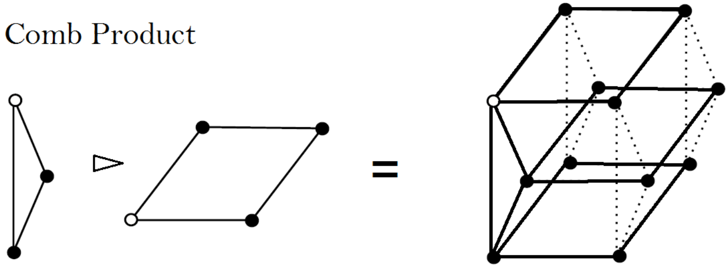
In $(\mathcal{M}_{nm}, \phi_1, Tr)$, the matrices $I_n \otimes A$ and $B \otimes P_m$ are monotone independent with respect to ϕ_1.



Cyclic monotone independence and comb product

Example

In $(\mathcal{M}_{nm}, \phi_1, Tr)$, the matrices $I_n \otimes A$ and $B \otimes P_m$ are cyclically monotone independent with respect to (ϕ_1, Tr).



Can we describe the distribution with respect to Tr of $I_n \otimes A$ and $B \otimes P_m$?

Cyclic monotone independence

We want the distribution w.r.t ω of $a+b$ when a and b are monotone ind. w.r.t φ and cylically monotone ind. w.r.t (φ,ω). To encode the moments w.r.t ω, we consider

$$\tilde{\mathfrak{g}}_x(z) = \sum_{n\geq 1} \frac{\omega(x^n)}{z^{n+1}}$$

Theorem

Let $(\mathcal{A},\varphi,\omega)$ be a cncps and $a,b\in\mathcal{A}$. Suppose that (a,b) is cyclic-monotone independent. We then have

$$\tilde{\mathfrak{g}}_{a+b}(z) = \tilde{\mathfrak{g}}_b(z) + F_b'(z)\,\tilde{\mathfrak{g}}_a(F_b(z)).$$

Note that we only need $\tilde{\mathfrak{g}}_b, F_b$ and $\tilde{\mathfrak{g}}_a$. F_a is irrelevant.

Cyclic monotone convolution

Proof:
We expand $(a+b)^n$ as before.

$$(a+b)^n = b^n + \sum_{\substack{k\geq 1 \\ q_1,q_2,\dots,q_{k+1}\geq 0 \\ q_1+\cdots+q_{k+1}=n-k}} b^{q_1} a b^{q_2} a \cdots a b^{q_{k+1}},$$

This time applying ω yields

$$\omega((a+b)^n) = \omega(b^n) + \sum_{\substack{k\geq 1 \\ q_1,q_2,\dots,q_{k+1}\geq 0 \\ q_1+\cdots+q_{k+1}=n-k}} \varphi(b^{q_1+q_{k+1}})\varphi(b^{q_2})\cdots\varphi(b^{q_k})\omega(a^k)$$

Note again, only terms of the form $\omega(b^n), \omega(a^k), \varphi(b^r)$ appear on RHS.

Multiplying the above identity by z^{-n-1} and taking the summation over n yields

$$\begin{aligned}\tilde{\mathfrak{g}}_{a+b}(z) &= \tilde{\mathfrak{g}}_b(z) + \sum_{k\geq 1} \sum_{q_1,\ldots,q_{k+1}\geq 0} \frac{\varphi(b^{q_1+q_{k+1}})}{z^{q_1+q_{k+1}}} \frac{\varphi(b^{q_2})}{z^{q_2}} \cdots \frac{\varphi(b^{q_k})}{z^{q_k}} \frac{\omega(a^k)}{z^{k+1}} \\ &= \tilde{\mathfrak{g}}_b(z) - z^2 G_b'(z) \sum_{k\geq 1} [zG_b(z)]^{k-1} \frac{\omega(a^k)}{z^{k+1}} \\ &= \tilde{\mathfrak{g}}_b(z) - \frac{G_b'(z)}{G_b(z)^2} \sum_{k\geq 1} G_b(z)^{k+1} \omega(a^k) \\ &= \tilde{\mathfrak{g}}_b(z) + F_b'(z)\, \tilde{\mathfrak{g}}_a(F_b(z)).\end{aligned}$$

where we used the identity

$$\sum_{m\geq 0, n\geq 0}^{\infty} \frac{\varphi(b^{m+n})}{z^{m+n}} = \sum_{k\geq 0}^{\infty} (k+1)\varphi(b^k) z^{-k} = -z^2 G_b'(z).$$

Construction of cyclically monotone independent family

- Let $H_i, i \in \mathbb{N}$, be *finite-dimensional* Hilbert spaces with distinguished unit vectors $\Omega_i \in H_i$ respectively.
- Let $P_i \colon H_i \to H_i$ be the orthogonal projection onto $\mathbb{C}\Omega_i$ and φ_i be the vector state on $B(H_i)$ defined Ω_i.
- Let $H = H_1 \otimes \cdots \otimes H_N$, $\Omega = \Omega_1 \otimes \cdots \otimes \Omega_N$ and φ be the vacuum state on $B(H)$ defined by Ω.
- We embed $B(H_i)$ into $B(H)$:

$$\sigma_i(A) = I_{H_1} \otimes \cdots \otimes I_{H_{i-1}} \otimes A \otimes P_{i+1} \otimes \cdots \otimes P_N.$$

For a cyclically alternating tuple $(i_1, \ldots, i_n) \in [N]^n$ and $A_k \in B(H_{i_k})$, if $p \in [n]$ is such that $i_{p-1} < i_p > i_{p+1}$ then

$$\mathrm{Tr}_H(\sigma_{i_1}(A_1) \cdots \sigma_{i_n}(A_n)) = \varphi_{i_p}(A_p)\, \mathrm{Tr}_H\left[\cdots \sigma_{i_{p-1}}(A_{p-1})\sigma_{i_{p+1}}(A_{p+1}) \cdots\right],$$

Construction of cyclically monotone independent family

Definition

Let $(\mathcal{A}, \varphi, \omega)$ be a cncps. An ordered family of $*$-subalgebras $\{\mathcal{A}_i\}_{i\in I}$ of $\mathcal{A}$ is said to be cyclic-monotone independent if

1. it is monotonically independent with respect to φ, that is, for any $n \geq 2$, any alternating tuple $(i_1, \ldots, i_n) \in I^n$ (namely $i_1 \neq \cdots \neq i_n$) and $a_k \in \mathcal{A}_{i_k}$, $k = 1, 2, \ldots, n$, if $p \in [n]$ is such that $i_{p-1} < i_p > i_{p+1}$ then

$$\varphi(a_1 \cdots a_n) = \varphi(a_p)\varphi(a_1 \cdots a_{p-1}a_{p+1} \cdots a_n);$$

2. for any $n \geq 2$, cyclically alternating tuple $(i_1, \ldots, i_n) \in I^n$ (namely $i_1 \neq \cdots \neq i_n \neq i_1$) and $a_k \in \mathcal{A}_{i_k}$, $k = 1, 2, \ldots, n$, if $p \in [n]$ is such that $i_{p-1} < i_p > i_{p+1}$ then

$$\omega(a_1 \cdots a_n) = \varphi(a_p)\omega(a_1 \cdots a_{p-1}a_{p+1} \cdots a_n).$$

Central limit theorem

Let $S_N = a_1 + \cdots a_N$, with $a = \sigma(a_i)$.

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$$\omega(b_N) = \operatorname{Tr}(a) + d \operatorname{Tr}(a) + \cdots + d^{N-1} \operatorname{Tr}(a) = [N]_d \operatorname{Tr}(a),$$

where $[N]_d = 1 + d + d^2 + \cdots + d^{N-1} = (1 - d^N)/(1 - d)$. So

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Theorem

If $S_N = a_1 + \cdots a_N$

$$\lim_{N\to\infty} d^{-N}\omega(S_N^k) = \sum_{\pi\in OP(k)} \frac{\omega(\pi)}{(d-1)^{|\pi|}}. \quad (1)$$

In particular CLT is NOT universal and normalization is different (d is the dimension of H_i).

Central limit theorem

Theorem

Let A_N be the adjacency matrix of the N-fold comb product of (K_2, o) with itself. Then the empirical eigenvalue distribution of A_N converges weakly to λ as $N \to \infty$.

Smyth (1980) defined the distribution function $L_+ : [0, \infty) \to [0, 1)$ characterized by the property that L_+ is strictly increasing, $L_+(0) = 0$ and

$$|2L_+(x) - 1| = L_+(|x - x^{-1}|), \qquad x > 0.$$

If λ_+ is the distribution associated with L_+, then λ is the symmetrization of λ_+. It is known λ^+ (and hence λ) has no atoms.

Central limit theorem

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If $m_n(\lambda) = \beta_n$ then

$$\beta_n = \sum_{\ell=0}^{n-1} \binom{n+\ell}{n-\ell} \frac{2n}{n+\ell} \beta_\ell, \qquad n \geq 1, \tag{2}$$

Sloane's A048286: 2,10,80,874,12092,202384,3973580...

Related work

- Collins, Leid and Sakuma (2022) find other models for monotone and cyclically monotone. Their models are also compatible with each other.
- The results of Shlyakhtenko (2018) on type B freeness and Random Matrices may be interpreted as asymptotic cyclically monotone independence.
- Cébron and Gilliers (2022) consider cyclic-conditional freeness and give random matrix models.

Thanks!