Atoms for polynomials in free variables

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CIMAT

UC Berkeley Probabilistic Operator Algebra Seminar
September 13, 2021

Comparison

Theorem

Let $X_1, \ldots, X_d$ and $Y_1, \ldots, Y_d$ be normal variables in tracial W^*-probability spaces with $X_1, \ldots, X_d$ being $*$-free and such that, for each $1 \leq i \leq d$, we have

$$\mu_{X_i}(\{\lambda\}) = \mu_{Y_i}(\{\lambda\}).$$

Then, for each selfadjoint polynomial P in d non-commuting variables in d non-commuting variables,

$$\mu_{P(X)}(\{\lambda\}) \leq \mu_{P(Y)}(\{\lambda\}).$$

Our aim is to use the above theorem to obtain as much information of the atoms for the **free case** as we can from **specific choices** for Y_i's.

Comparison

Let $X = (X_1, \ldots, X_d)$ be a tuple of $*$-free normal random variables. We define

$$\mathcal{X}_m := \{Y = (Y_1, \ldots, Y_d) \in M_m(\mathbb{C})^d : \mu^p_{X_k} \leq \mu^p_{Y_k}\}$$

and $\mathcal{X} := \coprod_{m=1}^{\infty} \mathcal{X}_m$.
By our main theorem

$$\mu^p_{P(X)} \leq \inf_{Y \in \mathcal{X}} \mu^p_{P(Y)}.$$

Proposition

Let $X = (X_1, \ldots, X_d)$ be a tuple of $*$-free normal random variables. Then for any $P \in \mathbb{C}\langle x_1, \ldots, x_d \rangle$

$$\mu^p_{P(X)} = \inf_{Y \in \mathcal{X}} \mu^p_{P(Y)}.$$

Example: Free additive convolution

Theorem (Bervocivi Voiculescu 98)

$\mu \boxplus \nu$ has an atom at $a \in \mathbb{R}$ if and only if there exist $\lambda, \rho \in \mathbb{R}$ such that $\rho + \lambda = a$ and $\mu(\lambda) + \nu(\rho) > 1$. Moreover, if $\mu(\lambda) + \nu(\rho) > 1$, we have $\mu \boxplus \nu(a) = \mu(\lambda) + \nu(\rho) - 1$.

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Lower bound: Simple and intuitive. Take matrices $X, Y \in M_n$.
If $dim(Ker(X - \lambda I)) = m = nt$ and $dim(Ker(Y - \rho I)) = l = ns$, then

$$\begin{aligned} dim(Ker(X + Y - \lambda I - \rho I)) &\geq dim(Ker(X - \lambda I) \cap Ker(Y - \rho I)) \\ &\geq (m + l - n) = n(t + s - 1) \end{aligned}$$

Upper bound: Case 1. $t + s > 1$. Consider the matrices,

$$X_n = \begin{pmatrix} \lambda & & & & & \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & \lambda_{i+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix}, Y_n = \begin{pmatrix} \rho_1 & & & & \\ & \ddots & & & \\ & & \rho_j & & \\ & & & \rho & & \\ & & & & \ddots & \\ & & & & & \rho \end{pmatrix}$$

where $i = sn \geq j = n - tn$.
Then

$$X_n + Y_n = \begin{pmatrix} \lambda + \rho_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda + \rho_j & & & & & & \\ & & & \lambda + \rho & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda + \rho & & & \\ & & & & & & \lambda_{i+1} + \rho & & \\ & & & & & & & \ddots & \\ & & & & & & & & \lambda_n + \rho \end{pmatrix}$$

Example: Free additive convolution

$$X_n + Y_n = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} \lambda + \rho_1 & & \\ & \ddots & \\ & & \lambda + \rho_i \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \lambda + \rho & & \\ & \ddots & \\ & & \lambda + \rho \end{pmatrix},$$

and

$$A_3 = \begin{pmatrix} \lambda_{j+1} + \rho & & \\ & \ddots & \\ & & \lambda_n + \rho \end{pmatrix}.$$

We see that the size of the eigenspace associated to a is $\dim(A_2) = i - j = (s + t - 1)n$.

Example: Free additive convolution

Upper bound: Case 1. $t + s < 1$.
We claim that we can reorder the eigenvalues in such a way that $\lambda_i + \rho_i \neq a$, for all i. Taking the matrices $X_n = diag(\lambda_1, \ldots, \lambda_n)$ and $Y_n = (\rho_1, \ldots, \rho_n)$, we see that $X_n + Y_n = diag(\lambda_1 + \rho_1, \ldots, \lambda_n + \rho_n)$ which has no eigenvalue equal to a, as desired.

Example: Free additive convolution

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Proof of claim: We prove this by induction on n. For $n = 1$ and $n = 2$, it is clear. Consider a reordering of $\{\lambda_i, \rho_i\}$ such $\lambda_i + \rho_i \neq a$ for $i \leq n - 1$ which is possible by induction. Now consider $\lambda_n + \rho_n$. If $\lambda_n + \rho_n \neq a$ we are done. Otherwise, if $\lambda_n + \rho_n = a$, then let

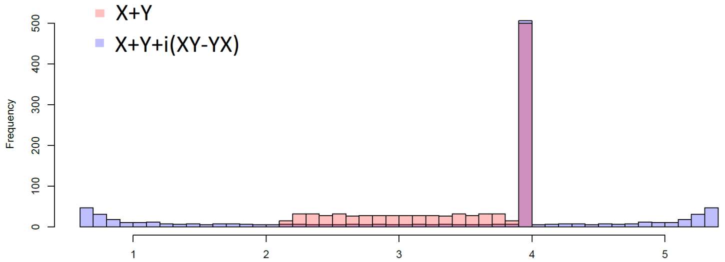
$$S = \{j \in [n-1] \mid \text{such that } \lambda_j = \lambda_n \text{ or } \rho_j = \rho_n\}.$$

If $|S| = n - 1$, by choosing for each j, ρ_j or λ_j, together with λ_n, ρ_n we have that $sn + tn \geq n + 1$, which yields a contradiction. Finally, if $|S| \leq n - 2$, there exists j such that $\lambda_j \neq \lambda_n$ and $\rho_j \neq \rho_n$. Since $\lambda_j + \rho_n \neq a$ and $\lambda_n + \rho_j \neq a$, we get the desired reordering.

(Injective) Polynomials in two variables

Theorem

Let X and Y be free and $a \in \mathbb{R}$. Let P satisfy that for all λ, ρ such that $P(\lambda, \rho) = a$ then $P(\lambda, \tilde{\rho}) \neq a$ and $P(\tilde{\lambda}, \rho) \neq a$ whenever $\lambda \neq \tilde{\lambda}$ and $\rho \neq \tilde{\rho}$.
Then $P(X, Y)$ has an atom at a if and only there are λ and ρ such that X has an atom at λ of size s and Y has an atom at ρ of size t, such that $r = t + s - 1 > 0$ and $P(\lambda, \rho) = a$.
Furthermore, if $r > 0$, and $s(a)$ and $t(a)$ denote the mass of this (unique) atoms, then the mass at a is given by $s(a) + t(a) - 1$.



Unavoidable atoms

Proposition

Let $(\mathcal{M}, \tau)$ *be a tracial* W^**-probability space and consider a tuple* $X = (X_1, ..., X_d)$ *of selfadjoint operators in* $\mathcal{M}$*. Suppose that for each* i*,* $\mu_i(\lambda_i) \geq t_i$ *for some* $\lambda_i \in \mathbb{R}$ *and* $t_i \in [0,1]$*. If* $t_1 + \cdots + t_d > d - 1$ *then for any selfadjoint polynomial* $P \in \mathbb{C}\langle x_1, \ldots, x_d \rangle$*, the measure* $\mu_{P(X)}$ *has an atom at* $P(\lambda)$ *of size at least* $t_1 + \cdots + t_d - (d-1)$*, where* $\lambda := (\lambda_1, \ldots, \lambda_d)$*.*

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Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space and consider a tuple $X = (X_1, ..., X_d)$ of selfadjoint operators in $\mathcal{M}$. Suppose that for each i, $\mu_i(\lambda_i) \geq t_i$ for some $\lambda_i \in \mathbb{R}$ and $t_i \in [0, 1]$. If $t_1 + \cdots + t_d > d - 1$ then for any selfadjoint polynomial $P \in \mathbb{C}\langle x_1, \ldots, x_d \rangle$, the measure $\mu_{P(X)}$ has an atom at $P(\lambda)$ of size at least $t_1 + \cdots + t_d - (d - 1)$, where $\lambda := (\lambda_1, \ldots, \lambda_d)$.

Proof

- For each i, let p_i be such $Xp_i = \lambda_i p_i$ for all $i \in \{1, \ldots, d\}$ and $\tau(p_i) = t_i$.
- Then for $p := \min(p_1, \ldots, p_d)$, we have $P(X)p = P(\lambda)p$.
- Finally $\tau(p) \geq \tau(p_1) + \cdots + \tau(p_d) - (d - 1) = t_1 + \cdots + t_d + d - 1$.

Unavoidable atoms

Proposition

Let $X, Y_1, ..., Y_d$ be selfadjoint operators in some tracial W^*-probability space $(\mathcal{M}, \tau)$. Suppose that $\mu_X(\{0\}) \geq t$. Then for any selfadjoint polynomial P of the form

$$P(x, y_1, ..., y_d) = \sum_{i=1}^{k} Q_{i,1}(x, y_1, ..., y_d) x Q_{i,2}(x, y_1, ..., y_d),$$

the analytic distribution of $P(X, Y_1, ..., Y_d)$ has an atom at 0 whose size is at least $\max(kt - (k-1), 0)$.

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Proof

$$\begin{aligned} rank(P(X, Y_1, \ldots, Y_d)) &\leq \sum_{i=1}^{k} rank(Q_{i,1}(X, Y_1, ..., Y_d) X Q_{i,2}(X, Y_1, ..., Y_d)) \\ &\leq k rank(X) = k(1-t), \end{aligned}$$

Injective Polynomials

Definition

Let $P \in \mathbb{C}\langle x_1, \ldots, x_d \rangle$ be given. For $a \in \mathbb{R}$, if $P(\rho_1, \ldots, \rho_i, \ldots, \rho_d) = a$ and

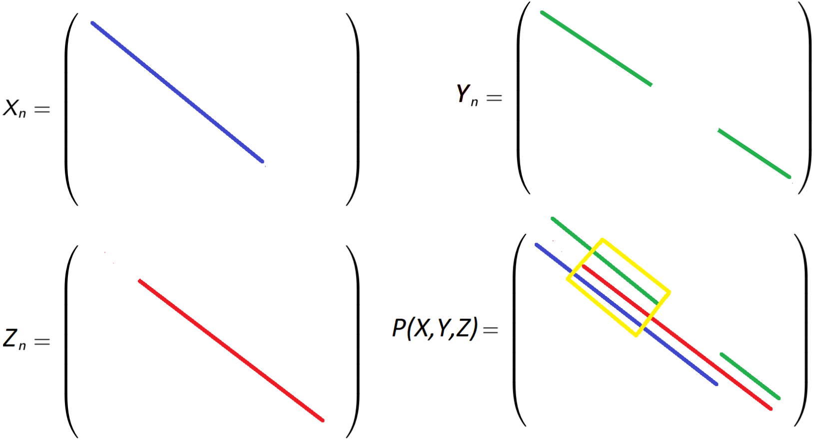
$$P(\rho_1, \ldots, \rho_i, \ldots, \rho_d) \neq P(\rho_1, \ldots, \tilde{\rho}_i, \ldots, \rho_d)$$

whenever $\rho_i \neq \tilde{\rho}_i$ for any given scalar values $\rho_1, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_d$, then we say P is *injective* for a.

Theorem

Let $P \in \mathbb{C}\langle x_1, \ldots, x_d \rangle$ be injective for $a \in \mathbb{R}$ and let $a = P(\lambda_1, \ldots, \lambda_d)$. Suppose that $X_1, \ldots, X_d$ are free selfadjoint random variables and $X_i \sim \mu_i$ with $\mu_i(\{\lambda_i\}) = t_i$. If $t_1 + \ldots + t_d \geq d - 1$ then the distribution of $P(X_1, \ldots, X_d)$ has an atom at a of size exactly $t_1 + \ldots + t_d - (d-1)$.

Idea of proof for 3 variables.



Not every polynomial is injective for all atoms

Theorem (Belinschi 2003)

If $\mu \in \mathcal{P}(\mathbb{R}^+)$ then $\mu \boxtimes \nu$ has an atom at $a \in \mathbb{R} \setminus \{0\}$ if and only if there exist $\lambda, \rho \in \mathbb{R}$ such that $\rho\lambda = a$ and $\mu(\lambda) + \nu(\rho) > 1$. Moreover

- *If $\mu(\lambda) + \nu(\rho) > 1$, we have $\mu \boxtimes \nu(a) = \mu(\lambda) + \nu(\rho) - 1$.*
- *The atom at 0 is given by $\mu \boxtimes \nu\{0\} = \max(\mu\{0\}, \nu\{0\})$.*

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Lower bound follows from directly (for instance from the second theorem for unavoidable atoms taking $k = 1$).

$\mu \boxtimes \nu\{0\} \leq \max(\mu\{0\}, \nu\{0\})$. We may assume that $\mu \sim X_n^2$ and $\nu \sim Y_n$. Consider X_n and Y_n,

$$X_n = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_m & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, Y_n = \begin{pmatrix} \rho_1 & & & & & \\ & \ddots & & & & \\ & & \rho_l & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

with $\lambda_i \neq 0$ for $0 \leq i \leq m$ and $\rho_j \neq 0$, for $0 \leq j \leq l$. Then, if $r = min(l, m)$, we have

$$X_n Y_n X_n = \begin{pmatrix} \lambda_1 \rho_1 \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_r \rho_r \lambda_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

The result follows since $Null(X_n Y_n X_n) = max\{Null(X_n), Null(X_n)\}$.

Commutative Case: Further consequences

Theorem

Let $X_1 \ldots, X_d$, be free. The possible atoms of $P(X_1, ..., X_d)$ are contained in the set

$$\{P(\rho_1, ..., \rho_d) \mid \rho_i \text{ is atom of } X_i,\ \forall i = 1, \ldots, d\}.$$

Furthermore, if $X_1, \ldots, X_d \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{R}$, then $\mu_{P(X_1,\ldots,X_d)}(\{\lambda\}) \leq k(\lambda)/n$, where

$$k(\lambda) := \min_{\sigma_1,\ldots,\sigma_d \in S_n} |\{j \in \{1, ...n\} : p(\lambda^{(1)}_{\sigma_1(j)}, ..., \lambda^{(d)}_{\sigma_d(j)}) = \lambda\}|,$$

where, for $i = 1, \ldots, d$, $\{\lambda^{(i)}_j\}_{j=1}^n$ denotes the eigenvalues of X_i.

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where, for $i = 1, \ldots, d$, $\{\lambda^{(i)}_j\}_{j=1}^n$ denotes the eigenvalues of X_i.
Example: $i(XY - YX)$ can only have atoms at 0. We'll see $k(0)$ is not an optimal bound.

2x2 matrices

Now we consider $X_n, Y_n \in M_{2n}(\mathbb{C})$ matrices consisting of diagonal blocks of size 2×2, $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ on their diagonals, respectively.

$$X_n = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}, Y_n = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}.$$

Similar as before

$$p(X_n, Y_n) = \begin{pmatrix} p(A_1, B_1) & 0 & \cdots & 0 \\ 0 & p(A_2, B_2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & p(A_n, B_n) \end{pmatrix}.$$

How to choose A_i and B_i?

Example: Commutator

Theorem

Let X and Y be free random variables. Let t and s be the size of the largest atom of X and Y, respectively, i.e.,

$$t = \max\{\mu_X(\{a\}) \mid a \in \mathbb{R}\} \qquad \textit{and} \qquad s = \max\{\mu_Y(\{b\}) \mid b \in \mathbb{R}\}.$$

Then

1. *$i(XY - YX)$ has an atom at 0 of size given by $max(2t - 1, 2s - 1, 0)$.*
2. *$i(XY - YX)$ has no futher atoms.*

Case $t, s \leq 1/2$**.** Our aim is to show that $i(XY - YX)$ has no atom at 0. We take $X_n, Y_n \in M_{2n}$ and reorder the eigenvalues so that $\lambda_{2i-1} \neq \lambda_{2i}$ and $\rho_{2i-1} \neq \rho_{2i}$

Now, consider the block-diagonal matrices

$$X_n = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix}, \qquad Y_n = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_n \end{pmatrix}$$

with

$$A_i = \begin{pmatrix} \lambda_{2i-1} & 0 \\ 0 & \lambda_{2i} \end{pmatrix} \qquad \text{and} \qquad B_i = \frac{1}{2} \begin{pmatrix} \rho_{2i-1} + \rho_{2i} & \rho_{2i-1} - \rho_{2i} \\ \rho_{2i-1} - \rho_{2i} & \rho_{2i-1} + \rho_{2i} \end{pmatrix}.$$

So

$$[X_n, Y_n] = \begin{pmatrix} [A_1, B_1] & & \\ & \ddots & \\ & & [A_n, B_n] \end{pmatrix}$$

and it is enough to prove that $[A_i, B_i]$ is invertible. But

$$[A_i, B_i] = \frac{1}{2} \begin{pmatrix} 0 & (\lambda_{2i} - \lambda_{2i-1})(\rho_{2i} - \rho_{2i-1}) \\ -(\lambda_{2i} - \lambda_{2i-1})(\rho_{2i} - \rho_{2i-1}) & 0 \end{pmatrix}.$$

whose determinant is $(\lambda_{2i} - \lambda_{2i-1})^2(\rho_{2i} - \rho_{2i-1})^2 \neq 0$.

$l = max(t, s) \geq 1/2$. Our aim is to show that $i(XY - YX)$ has atom at 0 of size at most $2l - 1$. Again we take $X_n, Y_n \in M_{2n}$ and in this case we may reorder the eigenvalues so that $\lambda_{2i-1} \neq \lambda_{2i}$ and $\rho_{2i-1} \neq \rho_{2i}$, for $i < n(2l - 1)$. Now the block-diagonal matrices are of the form

$$X_n = \begin{pmatrix} A_1 & & & & & \\ & \ddots & & & & \\ & & A_n & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \qquad Y_n = \begin{pmatrix} B_1 & & & & & \\ & \ddots & & & & \\ & & B_n & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

So

$$[X_n, Y_n] = \begin{pmatrix} [A_1, B_1] & & \\ & \ddots & \\ & & [A_n, B_n] \end{pmatrix}$$

and $[A_i, B_i]$ is invertible for $i < n(2l - 1)$.

Anticommutator

Theorem

Let X and Y be free random variables and let $Z = XY + YX$.
i)The size of the atom at 0 of Z is given by

$$l := \max\{2t-1, 2s-1, s+u-1, t+r-1, 0\},$$

where

1. *t is the size of the atom at 0 of X;*
2. *s is the size of the atom at 0 of Y;*
3. *u is the size of the largest atom outside of 0 of X;*
4. *r is the size of the largest atom outside of 0 of Y.*

ii) For any $a \neq 0$, Z has an atom at a if and only if there exist weights $s(a)$ and $t(a)$ such that $t(a) + s(a) - 1 > 0$, X has an atom at λ of size $s(a)$ and Y has an atom at ρ of size $t(a)$ and $2\lambda\rho = a$.
The size of the atom of Z at $a \neq 0$ is given by $\max\{s(a) + t(a) - 1, 0\}$.

The algebra of two non-commuting projections

We now want to consider $A_i, B_i \in M_2(\mathbb{C})$, in general position. Consider the projections

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} t & \sqrt{(1-t)t} \\ \sqrt{(1-t)t} & 1-t \end{pmatrix}.$$

And assume that

$$A_i = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = (\lambda_1 - \lambda_2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$B_i = \begin{pmatrix} t\mu_1 + \lambda_2(1-t) & (\mu_1 - \mu_2)\sqrt{(1-t)t} \\ (\mu_1 - \mu_2)\sqrt{(1-t)t} & \mu_1(1-t) + t\mu_2 \end{pmatrix}$$

$$= (\mu_1 - \mu_2) \begin{pmatrix} t & \sqrt{(1-t)t} \\ \sqrt{(1-t)t} & 1-t \end{pmatrix} + \mu_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In this way, $A = (\lambda_1 - \lambda_2)R + \lambda_2 I$ and $B = (\mu_1 - \mu_2)T + \mu_2 I$. So $P(A_i, B_i) = Q_i(R, T)$ for some polynomial Q_i.

The algebra of two non-commuting projections

The set of polynomials in R and T is well understood and rather easy to handle. Moreover, when considering T as a function of t, for all $t \in (0,1)$, the algebra generated by R and T is known to be isomorphic to the C^*-algebra generated by two non-commuting projections.
Since we have the relations:

$$RTR = tR, TRT = tT, T^2 = T \text{ and } R^2 = R,$$

any polynomial in R and T may be written as a linear combination

$$aR + bT + cRT + dTR + eI,$$

which actually has the matrix form

$$\begin{pmatrix} a+e+(b+c+d)t & (b+c)\sqrt{(1-t)t}+c \\ (b+d)\sqrt{(1-t)t} & e+b(1-t) \end{pmatrix}.$$

The determinant is $a(b(-t)+b+e)+be+cdt^2-cdt+cet+det+e^2$ which for $e=0$, reduces to $(t-1)(cdt-ab)$.

The algebra of two non-commuting projections

Definition

For $\lambda, \rho \in \mathbb{C}$, let us define define $Subs_{\lambda,\rho} : C\langle x, y\rangle \to \mathbb{C}[t]$ which sends $P \mapsto Subs_{\lambda,\rho}[P]$ as follows: For a monomial $m = m(x, y)$, $Subs_{\lambda,\rho}[m] \in \mathbb{C}[t]$, is given by the formula

$$Subs_{\lambda,\rho}[m](t) = m(\lambda, \rho)t^{k-1},$$

whenever $m(x, y)$ is of any of the following forms:
$m(x, y) = x^{i_1}y^{j_1}\cdots x^{i_k}, m(x, y) = y^{i_1}x^{j_1}\cdots y^{i_k}$,
$m(x, y) = x^{i_1}y^{j_1}\cdots y^{i_k}, m(x, y) = y^{i_1}x^{j_1}\cdots x^{i_k}$.
We extend $Subs_{\lambda,\rho}$ by linearity to any polynomial $P \in \mathbb{C}\langle x, y\rangle$ with no linear term.

Definition

Let $P \in \mathbb{C}\langle x, y\rangle$, be a polynomial with no constant term and write $P = P_1 + P_2 + P_3 + P_4$. where

$$P_1 = xQ_1y, \quad P_2 = yQ_2x, \quad P_3 = xQ_3x + ax \text{ and } \quad P_4 = yQ_4y + by$$

for some polynomials $Q_i \in \mathbb{C}\langle x, y\rangle$ and $a, b \in \mathbb{C}$.

1. We say that P satisfies the determinant condition on (λ, μ) if *for some* $t \in [0, 1]$,

$$tSubs_{\lambda,\rho}[P_1](t)Subs_{\lambda,\rho}[P_2](t) \neq Subs_{\lambda,\rho}[P_3](t)Subs_{\lambda,\rho}[P_4](t). \quad (1)$$

Some easy conditions which imply determinant condition for all λ, μ.

- $P_1 = 0$ or $P_2 = 0$ and $P_3, P_4 \neq 0$.
- $P_3 = 0$ or $P_4 = 0$ and $P_1, P_2 \neq 0$.
- $deg(P_1P_2) \neq deg(P_3P_4) + 1$.
- P_3 and P_4 have constant term.

Proposition

Let X and Y be free random variables and let $t \geq s$ be such that $t + s \geq 1$. Suppose that $X \sim \mu$ and $Y \sim \nu$ with $\mu = t\delta_0 + (1-t)\tilde{\mu}$ and $\nu = s\delta_0 + (1-s)\tilde{\nu}$, with $\tilde{\mu} = \sum_{i=1}^{l} \frac{1}{n}\delta_{\lambda_i}$ and $\tilde{\nu} = \frac{1}{n}\sum_{i=1}^{m} \delta_{\rho_i}$ with $\lambda_i \neq 0$ and $\rho_i \neq 0$. If there exist an injective function $\sigma : [l] \to [m]$ such that $P(x, y)$ satisfies the determinant condition for $\lambda_i, \rho_{\sigma(i)}$, then we have the following.

1. *The atom of $P(X, Y)$ at 0 has size at most $2t - 1$.*
2. *If for all ρ, $P(0, \rho) \neq 0$ then the atom at 0 has exactly size $s + t - 1$.*

Consider the block-diagonal matrices in M_{2n} given by

$$X_n = \begin{pmatrix} A_1 & & & & & \\ & \ddots & & & & \\ & & A_{2l} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, Y_n = \begin{pmatrix} B_1 & & & & & \\ & \ddots & & & & \\ & & B_{2m} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

where $A_i = \lambda_i R$ for $i = 1, \dots, 2l$, $B_i = \rho_i T$ for $i = 1, \dots, 2m$. We see that

$$P(X_n, Y_n) = \begin{pmatrix} P(A_1, B_1) & & & & & & & & \\ & \ddots & & & & & & & \\ & & P(A_{2l}, B_{2l}) & & & & & & \\ & & & P(0, B_{2l+1}) & & & & & \\ & & & & \ddots & & & & \\ & & & & & P(0, B_{2m}) & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix}$$

- If $P(A_i, B_i)$ is invertible, for all $1 \leq i \leq 2l$, then $P(X_n, Y_n)$ has nullity at most $2n - 4l = 2n(1 - 2(1 - t)) = 2n(2t - 1)$.
- A_i and B_i satisfy the relations

$$A_iB_iA_i = \lambda_i^2 \rho_i t A_i \ , \ B_iA_iB_i = \lambda_i \rho_i^2 t B_i \ , \ A_i^k = \lambda_i^k A_i \ and \ B_i^k = \rho_i^k B_i.$$

- We arrive at the following:

$$\begin{aligned} P(A_i, B_i) &= Subs[P_1(t, \lambda_i, \rho_i)]RT + Subs[P_2(t, \lambda_i, \rho_i)]TR \\ &+ Subs[P_3(t, \lambda_i, \rho_i)]R + Subs[P_4(t, \lambda_i, \rho_i)T. \end{aligned}$$

- For $a_1, a_2, a_3, a_4 \in \mathbb{C}$, the determinant of $a_1RT + a_2TR + a_3R + a_4T$ is $(t-1)(ta_1a_2 - a_3a_4)$. Thus $\det[P(A_i, B_i)] \neq 0$ for some t whenever the determinant condition holds for λ_i and ρ_i.

Thanks...