Spectral operators in finite von Neumann algebras

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The Spectral Theorem for Normal Operators

If $T \in B(\mathcal{H})$ is normal, then there exists a projection-valued measure E_T such that

$$T = \int_{\sigma(T)} \lambda \, dE_T(\lambda)$$

Spectral Measures

Let $\mathfrak{A}$ be the Borel σ-algebra of the complex plane $\mathbb{C}$. A *bounded projection-valued spectral measure* is a mapping $B \mapsto E(B)$ that assigns to every $B \in \mathfrak{A}$ a projection $E(B) \in B(\mathcal{H})$ $(E(B)^2 = E(B), E(B)^* = E(B))$, so that

(i) $E(\mathbb{C}) = 1$,

(ii) for all $B_1, B_2 \in \mathfrak{A}$, $E(B_1 \cap B_2) = E(B_1)E(B_2)$,

(iii) for all $B_1, B_2, \ldots \in \mathfrak{A}$ such that $B_i \cap B_j = \emptyset$ whenever $i \neq j$,

$$E(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} E(B_i),$$

where the sum converges in the strong operator topology.

Spectral Measures

Definition

Let $\mathfrak{A}$ be the Borel σ-algebra of the complex plane $\mathbb{C}$. A bounded **idempotent**-*valued spectral measure is a mapping $B \mapsto E(B)$ that assigns to every $B \in \mathfrak{A}$ an* **idempotent** $E(B) \in B(\mathcal{H})$ $(E(B)^2 = E(B))$*so that*

(i) $E(\mathbb{C}) = 1$,

(ii) *for all* $B_1, B_2 \in \mathfrak{A}$, $E(B_1 \cap B_2) = E(B_1)E(B_2)$,

(iii) *for all* $B_1, B_2, \ldots \in \mathfrak{A}$ *such that* $B_i \cap B_j = \emptyset$ *whenever* $i \neq j$,

$$E(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} E(B_i),$$

where the sum converges in the strong operator topology.

(iv) $\sup_{B \in \mathfrak{A}} \|E(B)\| < \infty$.

Spectral measures

Dunford: an operator T is called a spectral operator if there exists an (idempotent-valued) spectral measure E such that

- $E(B)T = TE(B)$, for every Borel set B. (in particular, $E(B)\mathcal{H}$ is an invariant subspace for T)
- The spectrum of T restricted to the range of $E(B)$ is contained in $\overline{B}$

An operator S is said to be of *scalar type* if S is spectral, and satisfies the equation

$$S = \int_{\sigma(S)} \lambda E(d\lambda)$$

Proposition

Suppose E is a bounded idempotent valued spectral measure in a von Neumann algebra $\mathcal{M}$. Then there is an invertible $A \in \mathcal{M}$ so that for every Borel B, the idempotent $A^{-1}E(B)A$ is self-adjoint.

Theorem (Dunford, '58)

$S \in B(\mathcal{H})$ is a scalar type operator iff there exists A invertible in $B(\mathcal{H})$, such that $A^{-1}SA$ is a normal operator. Moreover, $A \in \{S\}''$

Theorem (Dunford, '58)

T is a spectral operator iff it is similar to the sum of a commuting normal operator and a quasinilpotent operator.

(analogous to the Jordan Canonical Form for matrices)

Decomposable operators

Definition (Foais)

An operator $T \in B(\mathcal{H})$ is said to be decomposable if it has a spectral capacity. A spectral capacity for T is a map

$$\{\text{closed sets in } \mathbb{C}\} \ni K \mapsto \mathcal{E}(K) \in \{\text{closed subspaces of } \mathcal{H}\}$$

so that

1. $\mathcal{E}(\emptyset) = \{0\}$, $\mathbb{C} = \mathcal{H}$
2. $\mathcal{E}(\overline{U_1}) + \mathcal{E}(\overline{U_2}) + ... + \mathcal{E}(\overline{U_n}) = \mathcal{H}$ *(algebraic sum) for all open covers* $\{U_1, ...U_n\}$ *of* $\mathbb{C}$
3. $E(\cap_{k=1}^{\infty} K_n) = \cap_{k=1}^{\infty} E(K_n)$
4. $\sigma(T\restriction_{\mathcal{E}(K)}) \subseteq K$

If T is spectral with spectral measure E, then T is decomposable with spectral capacity $\mathcal{E}(K) = E(K)\mathcal{H}$

Brown measure

Let $\mathcal{M}$ be a finite von Neumann algebra with trace τ.
For a normal operator $T \in \mathcal{M}$, there is a complex-valued measure associated with the projection-valued measure, given by

$$\mu_T(B) = \tau(E_T(B))$$

Theorem (Brown,'83)

Let $T \in \mathcal{M}$. Then there exists a unique probability measure μ_T such that for every $\lambda \in \mathbb{C}$,

$$\int_{[0,\infty)} \log(x) d\mu_{|T-\lambda|}(x) = \int_{\mathbb{C}} \log|z-\lambda| d\mu_T(z),$$

for $S \geq 0$, $\mu_S =$ spectral distribution measure $\tau \circ E$, where E is the spectral measure for S.
The measure μ_T is called the *Brown measure* of T. If T is normal, μ_T equals the spectral distribution measure of T.

Haagerup-Schultz projections

Haagerup and Schultz constructed a set of invariant projections for T, which behave well with the Brown measure:

Theorem (Haagerup, Schultz '09)

Let $T \in \mathcal{M}$. For any Borel set $B \subset \mathbb{C}$, there exists a unique projection $p = P(T, B) \in \mathcal{M}$ such that

(i) $Tp = pTp$

(ii) $\tau(p) = \mu_T(B)$

(iii) *when $p \neq 0$, considering pTp as an element of $p\mathcal{M}p$, its Brown measure is concentrated in B.*

(iv) *when $p \neq 1$, considering $(1-p)T(1-p)$ as an element of $(1-p)\mathcal{M}(1-p)$, $\mu_{(1-p)T}$ is concentrated in $\mathbb{C} \setminus B$.*

(v) *If $q \in \mathcal{M}$ is a T-invariant projection with $\mu_{q\mathcal{M}q}(Tq)$ concentrated in B, then $q \leq p$.*

Moreover, $P(T, B)$ is T-hyperinvariant, and if $B_1 \subset B_2$, then $P(T, B_1) \leq P(T, B_2)$.

Haagerup-Schultz projections

Proposition (Haagerup,Schultz)

If T is decomposable with spectral capacity $\mathcal{E}$, then $\mathcal{E}(K) = P(T, K)\mathcal{H}$. Moreover, the support of μ_T is equal to $\sigma(T)$.

Proposition

Let $T \in \mathcal{M}$ be a spectral operator with idempotent valued spectral measure E. Then, for every Borel set B,

$$P(T, B)\mathcal{H} = E(B)\mathcal{H} \tag{1}$$

Proof depends on the fact that $P(T, .)$ is a lattice map.

Theorem (Haagerup, Schultz '09)

$\mu_T = \delta_0$ *iff*
$\lim\left((T^n)^* T^n\right)^{1/n} \to 0$ *in the strong operator topology*
Such operators are called s.o.t-quasinilpotents.

HS projections for commuting tuples

Joint Brown measures and Haagerup–Schultz projections can also be defined for commuting tuples of operators.

Theorem (Schultz)

Let $S, T \in \mathcal{M}$ be commuting operators. Then, there exists a unique compactly supported Borel probability measure $\mu_{S,T}$ on $\mathbb{C}^2$ such that

$$\tau(\log|\alpha S + \beta T - 1|) = \int_{\mathbb{C}^2} \log|\alpha z + \beta w - 1| d\mu_{S,T}(z, w).$$

Theorem (Charlesworth,Dykema,Sukochev,Zanin)

For commuting operators $S, T \in \mathcal{M}$*, and a Borel set* $B \subset \mathbb{C}^2$*, there is a projection* $P((S,T) : B) \in \mathcal{M}$ *which is* (S,T)*-hyperinvariant, and which satisfies the following:*

(i) *For* $B_1, B_2 \subset \mathbb{C}$*,* $P((S,T) : B_1 \times B_2) = P(S, B_1) \wedge P(T, B_2)$

(ii) $P((S,T) : \cdot)$ *is a lattice map.*

(iii) *For a Borel set* B*, with* $p = P((S,T) : B)$*, the Brown measure of* (Sp, Tp) *and* $((1-p)S, (1-p)T)$ *, computed in the compressions* $p\mathcal{M}p$ *and* $(1-p)\mathcal{M}(1-p)$ *respectively, are concentrated in* B*, and* B^c*.*

(iv) $\mu_{(S,T)}(B) = \tau(P((S,T) : B))$.

The joint Brown measures and Haagerup–Schultz projections behave well under pushforwards. In particular,

Proposition

Let $S, T \in \mathcal{M}$ *be commuting operators. Let* $a : \mathbb{C}^2 \to \mathbb{C}$ *denote the addition map. Then, for any Borel set* $B \subset \mathbb{C}$*, we have*

$$P(S + T, B) = P((S, T) : a^{-1}(B)).$$

Hence, if T *is s.o.t.-quasinilpotent, then* $P(S + T, B) = P(S, B)$*.*

Angles between subspaces

Definition

Let V, W be closed non-zero subspaces of a Hilbert space $\mathcal{H}$, with $V \cap W = \{0\}$. Then, the angle between them is

$$\alpha(V, W) := \inf \left\{ \cos^{-1}\left(|\langle v, w \rangle|\right) \mid v \in V,\, w \in W,\, ||v|| = ||w|| = 1 \right\}$$

If p and q are projections in $B(\mathcal{H})$ with $p \wedge q = 0$, then we let $\alpha(p, q) = \alpha(p\mathcal{H}, q\mathcal{H})$.

We say that $T \in \mathcal{M}$ has the **uniformly nonzero angle property** (or UNZA property), if there is $\kappa > 0$ such that for all Borel sets $B \subseteq \mathbb{C}$ with $P(T, B) \neq 0 \neq P(T, B^c)$, we have

$$\alpha(P(T, B), P(T, B^c)) \geq \kappa$$

Question: Does NZA imply UNZA?

UNZA

UNZA allows us to construct lots of idempotents:

Lemma

Let V, W be closed subspaces of $\mathcal{H}$ with $V \cap W = \{0\}$ and $\overline{V + W} = \mathcal{H}$. Then the following are equivalent:

(i) $\alpha(V, W) > 0$.

(ii) $V + W$ *is closed.*

(iii) *There exists a bounded idempotent* $e \in B(\mathcal{H})$ *such that*

$$e\mathcal{H} = V \quad \text{and} \quad (1 - e)\mathcal{H} = W.$$

Moreover,there is a continuous, strictly decreasing function $f : (0, 1] \to [1, \infty)$ *such that*

$$\|e\| \leq f\big(1 - \cos(\alpha(V, W))\big).$$

Lemma

Assume Let $T \in \mathcal{M}$ *has the uniformly non-zero angle property. Then there exists an idempotent valued spectral measure* E *with the following properties,*

(a) $E(B)\mathcal{H} = P(T, B)\mathcal{H}$ *and* $\ker E(B) = P(T, B^c)\mathcal{H}$,

(b) $TE(B) = E(B)T$,

(c) *The Brown measure of the restriction of* T *to the range of* $E(B)$ *is concentrated in* B.

where B *is an arbitrary Borel subset of* $\mathbb{C}$.

Theorem (Dykema, K.U.)

Let $T \in \mathcal{M}$. Then the following are equivalent:

(a) *T has the UNZA property,*

(b) *there exist $S, Q \in \mathcal{M}$ with $[S, Q] = 0$, S a scalar type operator and Q s.o.t.-quasinilpotent, such that $T = S + Q$,*

(c) *there exist $A, N, Q' \in \mathcal{M}$, with $[N, Q'] = 0$, N normal, Q' s.o.t-quasinilpotent, and A invertible, such that $ATA^{-1} = N + Q'$.*

Proof: (a) $\implies$ (b)
T has an associated idempotent valued spectral measure E. Let

$$S = \int \lambda dE(\lambda)$$

S is a scalar type operator.
Claim: $Q = T - S$ is s.o.t-quasinilpotent.
For every Borel set $B \subseteq \mathbb{C}$,

$$P(S, B)\mathcal{H} = E(B)\mathcal{H} = P(T, B)\mathcal{H},$$

Since $TS = ST$,

$$P(Q, B) = P((T, S) : \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : \lambda_1 - \lambda_2 \in B\})$$

$P((T, S) : B_1 \times B_2) = P(T, B_1) \wedge P(T, B_2)$ so

$$\mathrm{supp}(\mu_{(T,S)}) \subseteq \{(z, z) : z \in \mathbb{C}\}$$

Hence, if $0 \notin B$, then $P(Q, B) = 0$.

(b) $\implies$ (c): Similar to Wermer/Dunford's construction, uses

$$\mu_{ATA^{-1}} = \mu_T$$

(c) $\implies$ (a):
If UNZA fails, there exist sets B_n so that

$$\alpha\left(P(T, B_n), P(T, B_n^c)\right) \to 0$$

Choose $v_n \in P(T, B_n)\mathcal{H}$, $w_n \in P(T, B_n^c)\mathcal{H}$ such that $\langle v_n, w_n \rangle \to 1$, and $||v_n|| = ||w_n|| = 1$.
Since N is normal, its spectral subspaces are orthogonal, and

$$P(N, B)\mathcal{H} = P(ATA^{-1}, B)\mathcal{H} = AP(T, B)\mathcal{H} \qquad (2)$$

So

$$||A(v_n - w_n)||^2 = ||Av_n||^2 + ||Aw_n||^2 \geq 2\left|\left|A^{-1}\right|\right|^{-2} > 0, \qquad (3)$$

which contradicts the fact that $||v_n - w_n|| \to 0$.

Corollary

Let $T \in \mathcal{M}$. Then, T is spectral if and only if T is decomposable and satisfies the UNZA property.

Corollary

Let $T \in \mathcal{M}$. Then $P(T, \cdot)$ defines a spectral measure if and only if $T = N + Q$ for some $N, Q \in \mathcal{M}$, where N is normal, Q is s.o.t.-quasinilpotent, and $NQ = QN$.

Non-spectral but decomposable operator

Let

$$T = \bigoplus_{n=1}^{\infty} \begin{pmatrix} 0 & 1 \\ 0 & 1/n \end{pmatrix} \in \bigoplus_{n=1}^{\infty} M_2(\mathbb{C}) \subseteq \mathcal{R}$$

$$\sigma(T) = \{1, 1/2, 1/3 \ldots\} \cup \{0\}$$

The eigenvectors for the nth block are $(1,0)^t$ and $(1,1/n)^t$, and the angle between them goes to 0 and $n \to \infty$.
But T has countable spectrum, so it is decomposable.

Non-trivial examples of decomposable, non-spectral operators in a finite factor?

R-diagonal operators

$x \in (\mathcal{M}, \tau)$ is R-diagonal if it has the same $*$-distribution as uh, where u is a Haar unitary, h is positive, and the pair (u, h) is $*$-free.

Proposition

Suppose $x \in \mathcal{M}$ is R-diagonal.

(i) *if x is invertible, then also x^{-1} is R-diagonal,*

(ii) *for every $k \in \mathbb{N}$, $\|x^k\|_2^2 = \|x\|_2^{2k}$,*

(iii) *the spectral radius of x equals $\|x\|_2$.*

(iv) *supp$(\mu_x) = \sigma(x)$, and μ_x is radially symmetric.*

DT-operators

Theorem (Dykema,Haagerup)

Let μ *be a compactly supported Borel probability measure on* $\mathbb{C}$. *For* $N \in \mathbb{N}$, *let*

$$D_N = diag(d_1^{(N)}, d_2^{(N)}, ..., d_N^{(N)})$$

$$T_N = \begin{pmatrix} 0 & G_{12} & \cdots & \cdots & G_{1N} \\ 0 & 0 & G_{23} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & 0 & G_{N-1,N} \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

with d_i *iid-*μ *distributed random variables,* G_{ij} *independent Gaussian*$(0, 1/N)$. *Then the pair* D_n, T_n *converge jointly in* $*$*-moments as* $n \to \infty$.

Definition

$c > 0$. Z is a $DT(\mu, c)$ operator if Z has the same $*$-distribution as the limiting $*$-distribution of $D_N + cT_N$

For each compactly supported Borel probability measure μ on $\mathbb{C}$ and each $c > 0$, there is a $DT(\mu, c)$ operator Z,

Fact:Can be realized as $Z = D + cT$, where D is a normal operator and T is the "upper triangular half" of a semicircular operator that is free from an abelian algebra containing D.

Upper triangular forms for DT operators

Theorem (Dykema,Haagerup)

If

- $(a_k)_{k=1}^N$, $(b_{ij})_{1\le i<j\le N}$ *a* $*$*-free family in* $(\mathcal{M},\tau)$
- *each* b_{ij} *circular with* $\tau(b_{ij}^*b_{ij}) = \frac{1}{N}$,
- *each* a_j *a* $DT(\mu_j, \frac{1}{\sqrt{N}})$ *operator for a Borel probability measure* μ_j *on* $\mathbb{C}$,
- $\frac{1}{N}(\mu_1 + \cdots + \mu_N) = \mu$

Then

$$Z = \begin{pmatrix} a_1 & b_{12} & \cdots & \cdots & b_{1N} \\ 0 & a_2 & b_{23} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & a_{N-1} & b_{N-1,N} \\ 0 & \cdots & \cdots & 0 & a_N \end{pmatrix},$$

is a DT(μ*,1) operator.*

Circular free Poisson operators

A circular free Poisson operator of parameter $c \geq 1$ is an R-diagonal operator x with $|x|^2$ has moments equal to those of a free Poisson distribution ν_c with parameter c.
This distribution is absolutely continuous with respect to Lebesgue measure and has density

$$\frac{d\nu_c}{d\lambda}(t) = \frac{\sqrt{(b-t)(t-a)}}{2\pi t} 1_{[a,b]}(t),$$

where $a = (\sqrt{c} - 1)^2$ and $b = (\sqrt{c} + 1)^2$.

Proposition

If x is a circular free Poisson with parameter $c \geq 1$, μ_x is the uniform distribution on the annulus $A_{\sqrt{c-1},\sqrt{c}}$.

Fact: The DT-operators that are also R-diagonal are precisely scalar multiples of the circular free Poisson operators

Haagerup-Schultz projections

Proposition (Haagerup,Schultz)

Let $r > 0$. Suppose $\mathcal{M}$ acts on the Hilbert space $\mathcal{H}$ and $T \in \mathcal{M}$. Then

$$P(T, r\mathbb{D})\mathcal{H} =$$
$$\{\xi \in \mathcal{H} \mid \exists \xi_n \in \mathcal{H}, \lim_{n\to\infty} \|\xi_n - \xi\| = 0, \limsup_{n\to\infty} \|T^n \xi_n\|^{1/n} \leq r\}.$$

and

$$P(T, \mathbb{C} \setminus r\mathbb{D})\mathcal{H} =$$
$$\{\eta \in \mathcal{H} \mid \exists \eta_n \in \mathcal{H}, \lim_{n\to\infty} \|T^n \eta_n - \eta\| = 0, \limsup_{n\to\infty} \|\eta_n\|^{1/n} \leq \frac{1}{r}\}.$$

Angle Estimates for DT operators

Lemma

Let $0 < r' < r < s < s'$ *so that the annuli* $A(r', r)$ *and* $A(s, s')$ *have the same Lebesgue measure. Let* T *be a DT-operator with uniform measure* μ *on* $A(r', r) \cup A(s, s')$*.*
Then

$$\cos(\alpha(P(T, A(r', r)), P(T, A(s, s')))) \geq \frac{1}{\sqrt{2(s^2 - r^2) + 1}}$$

Proof: Let μ_1 and μ_2 denote the uniform measure on $A(r', r)$ and $A(s, s')$ respectively.
There exist DT operators $D_1, D_2 \in \mathcal{M}$ and a circular operator $Z_1 \in \mathcal{M}$ so that D_i has Brown measure μ_i, $\tau(Z_1^* Z_1) = 1/2$, D_1, D_2, Z_1 are *-free, and

$$Z = \begin{pmatrix} D_1 & Z_1 \\ 0 & D_2 \end{pmatrix}$$

is a DT operator with measure μ in $(M_2(\mathcal{M}), \tau \circ \mathrm{tr}_2)$, acting on $L^2(\mathcal{M}) \oplus L^2(\mathcal{M})$.

Let

$$\eta_n = D_2^{-n} 1_{\mathcal{M}}$$

$$\xi_n = Z_1 D_2^{-n-1} 1_{\mathcal{M}}$$

Then

$$Z^n \binom{\xi_n}{\eta_n} = \begin{pmatrix} \sum_{k=0}^{n} D_1^k Z_1 D_2^{-k-1} 1_{\mathcal{M}} \\ 1_{\mathcal{M}} \end{pmatrix}$$

Let

$$\xi = \sum_{k=0}^{\infty} D_1^k Z_1 D_2^{-k-1}, \qquad \eta = 1$$

Estimate 2-norm of ξ using R-diagonality and $*$-freeness.

We get

$$(\xi, 1) \in P(Z, B(0, s)^c)$$

$$(\xi, 0) \in L^2(\mathcal{M}) \oplus 0 = P(Z, B(0, r))\mathcal{H}$$

Estimate angle between $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ and $\begin{pmatrix} \xi \\ 0 \end{pmatrix}$ to get the bound.

Theorem (Dykema, K.U.)

Let Z be a circular free Poisson operator with parameter $c \geq 1$. Then Z does not have the uniformly non-zero angles property, and hence is not spectral.

Proof sketch:

- Partition the annulus $A_{\sqrt{c-1},\sqrt{c}}$ into N equally weighted mutually disjoint parts $E_1 ... E_N$, with E_1, E_2 equally weighted annuli.
- Use the upper triangular decomposition, and the previous trick to get angle estimates.
- Choose E_1, E_2 to be sufficiently close, to show the UNZA property fails.

Extensions

Theorem

Let μ be a measure on $\mathbb{C}$ which is radially symmetric such that its support is not concentrated on a finite union of circles. Let Z be a $DT(\mu, c)$ operator. Then Z does not have the UNZA property, and hence is not spectral.

- Use fact that $Z = D + T$, with T a B-valued circular operator, to replace 2-norm estimates coming from R-diagonality.
- Generalize upper triangular decomposition of DT-operators to include convex combinations

$$\mu = t_1\mu_1 + ... + t_n\mu_n$$

where $0 < t_i < 1$, $\sum t_i = 1$, and t_i are not necessarily rational.