Convergence for non-commutative rational functions evaluated in random matrices

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Joint work with Tobias Mai, Benoît Collins, Felix Parraud and Sheng Yin

Today's talk

1. Non-commutative rational functions
   - Evaluation of non-commutative rational functions.
   - Theorem by Mai, Speicher, and Yin.
2. Main results
   - Main result 1: Well-definedness of $r(X^N)$ with $N \times N$ random matrices X^N.
   - Main result 2: Convergence in distribution.
3. Strategy of the proof
   - Linearization.
   - Characterization of cumulative distribution functions by projections.

Non-commutative rational function

Non-commutative rational expressions

- Non-commutative (nc) rational expressions are defined by all possible combinations of $x_1, \dots, x_d, \mathbb{C}$ with $+, \times, \cdot^{-1}, ()$.
  e.g. $x_1 x_2^{-1}$, $(x_1 + x_2)^{-1}$, $x_1 + 2x_2^{-1}x_1$
- For unital $\mathbb{C}$-algebra $\mathcal{A}$ and a nc rational expression r, we define

$$\mathrm{dom}_{\mathcal{A}}(r) = \{X = (X_1, \dots, X_d) \in \mathcal{A}^d : r(X) \in \mathcal{A}\}.$$

  For example, $\mathrm{dom}_{\mathcal{A}}((x_1x_2 - x_2x_1)^{-1}) = \emptyset$ when $\mathcal{A}$ is commutative.

Non-commutative rational functions

- For a nc rational expression r, $\mathrm{dom}(r)$ is a subset of **all square matrices** over $\mathbb{C}$ where evaluation of r is well-defined.
- **Equivalence relation**

$$r_1 \sim r_2 \Leftrightarrow r_1(a) = r_2(a), {}^{\forall} a \in \mathrm{dom}(r_1) \cap \mathrm{dom}(r_2) \neq \emptyset.$$

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- An equivalence class of nc rational expressions is called a non-commutative rational function.
- A set $\mathbb{C}\langle\!\langle x_1, \ldots, x_d \rangle\!\rangle$ of nc rational functions is called the free (skew) field which contains non-commutative polynomials $\mathbb{C}\langle x_1, \ldots, x_d\rangle$ (Amitur'66, Cohn'94, Kaliuzhnyi-Verbovetskyi and Vinnikov'10).

Example: NC rational expressions and the equivalence relation

$r_1 = (x_1(x_1+x_2)^{-1})x_2$, $r_2 = x_1((x_1+x_2)^{-1}x_2)$,
$r_3 = (x_1^{-1}+x_2^{-1})^{-1}$ are formally different rational expressions.

$$\begin{aligned} \mathrm{dom}(r_1) &= \mathrm{dom}(r_2) = \{(X_1,X_2); \det(X_1+X_2) \neq 0\} \\ \mathrm{dom}(r_3) &= \{(X_1,X_2); \det(X_1), \det(X_2), \det(X_1^{-1}+X_2^{-1}) \neq 0\}. \end{aligned}$$

We can see $\mathrm{dom}(r_3) \subsetneq \mathrm{dom}(r_2) = \mathrm{dom}(r_1)$ and r_i's are equivalent since we have the formal calculation,

$$\{x_1(x_1+x_2)^{-1}x_2\}^{-1} = x_2^{-1}(x_1+x_2)x_1^{-1} = x_1^{-1}+x_2^{-1}.$$

Remark

For any rational expression r with $\mathrm{dom}(r) \neq \emptyset$, there exists $N_0 = N_0(r)$ such that $\mathrm{dom}(r) \cap M_N(\mathbb{C})^d \neq \emptyset$ for $N \geq N_0$.

Evaluation of non-commutative rational functions

- We need to take matrices with large sizes for the evaluation of a nc rational expression.
- (Hall) For any $X_i \in M_2(\mathbb{C})$,

$$[[X_1, X_2]^2, X_3] = 0.$$

- (Amitsur-Levitzki) For any $X_i \in M_N(\mathbb{C})$,

$$\sum_{\pi \in S_{2N}} \mathrm{sgn}(\pi) X_{\pi(1)} \dots X_{\pi(2N)} = 0.$$

- For a nc rational function r, we would like to define

$$\mathrm{dom}_{\mathcal{A}}(r) = \bigcup_{r';[r']=r} \mathrm{dom}_{\mathcal{A}}(r'), \; r(X) = r'(X), \; X \in \mathrm{dom}_{\mathcal{A}}(r) \cap \mathrm{dom}_{\mathcal{A}}(r').$$

Evaluation in operators

- Evaluation of non-commutative rational functions in elements in a unital algebra $\mathcal{A}$ is not well-defined in general. For example, we have $x_1(x_2x_1)^{-1}x_2 = 1$, but for the unilateral shift S
$$S(S^*S)^{-1}S^* = SS^* \neq 1.$$
- Evaluation of non-commutative rational functions is well-defined if $\mathcal{A}$ is stably finite. i.e. we have for each $m \in \mathbb{N}$
$$A, B \in M_m(\mathcal{A}),\ AB = I_m \Leftrightarrow BA = I_m.$$
- Every finite von Neumann algebras $\mathcal{M}$ are stably finite.
- The $*$-algebra $\widetilde{\mathcal{M}}$ of affiliated operators with $\mathcal{M}$ is also stably finite.

Evaluation of all non-commutative rational functions

Theorem (T.Mai, R.Speicher and S.Yin '19)

Let $X = (X_1, \ldots, X_d)$ be a tuple of freely independent self-adjoint operators in a W^*-probability space such that each X_i has no atom. Then $^{\exists!}\mathrm{Ev}_s : \mathbb{C}\langle\!\langle x_1, \ldots, x_d \rangle\!\rangle \to \widetilde{\mathcal{M}}$, a homomorphism which extends $\mathbb{C}\langle x_1, \ldots, x_d\rangle \ni P \to P(X) \in \mathcal{M}$.

- The condition (free + absence of atom) is generalized to maximality of $\Delta(X_1, \ldots, X_d)(= d)$ defined in Connes-Shlyakhtenko'05

$$\dim_{\mathcal{M}\otimes\mathcal{M}^{\mathrm{op}}} \overline{\left\{ (T_1, \ldots, T_d) \in \mathcal{F}(L^2(\mathcal{M})) : \sum_{i=1}^{d} [T_i, JX_iJ] = 0 \right\}}^{\mathrm{HS}}.$$

- Free Haar unitaries $u_1, \ldots, u_d$ satisfy $\Delta(u_1, \ldots, u_d) = d$.

Remarks

- Atoms of a nc rational function evaluated in free random variables can be computed algebraically (Mai-Speicher-Yin'19, Arizmendi-Cébron-Speicher-Yin'21).
- The weight of atoms of a nc rational function evaluated in free random variables is minimal when each distribution is given (Arizmendi-Cébron-Speicher-Yin'21).
- nc rational functions are characterized by finite rank commutators, which is an analogue of Kronecker's theorem (Duchamp-Reutenauer'97, Linnell'00, M'22).

Main results

Asymptotic freeness of independent GUE's

Theorem (D.Voiculescu, 1991)

For independent GUE random matrices $X_1^N, \ldots, X_d^N$, we have almost surely,

$$\lim_{N\to\infty} \operatorname{tr}(X_{i_1}^N \cdots X_{i_n}^N) = \tau(s_{i_1} \cdots s_{i_n}),$$

where $s_1, \ldots, s_d$ are free semicircles with respect to τ.

- Let $P \in \mathbb{C}\langle x_1, \ldots, x_d \rangle$ be a self-adjoint polynomial. Then We have almost surely for $f \in C_c(\mathbb{R})$,

$$\lim_{N\to\infty} \int_{\mathbb{R}} f \; d\mu_{P(X^N)} = \tau[f(P(s))] = \int_{\mathbb{R}} f \; d\mu_{P(s)}.$$

- We replace nc polynomials by nc rational functions.

Main result 1: Evaluation in random matrices

- We work on rational functions evaluated in self-adjoint matrices and unitary matrices.

Theorem (Collins-Mai-M-Parraud-Yin'22)

Let r be a nc rational function with $d = d_1 + d_2$ formal variables. Let (X^N, U^N) be a tuple of random matrices in $M_N(\mathbb{C})^{d_1}_{\mathrm{sa}} \times U_N(\mathbb{C})^{d_2}$ whose law is absolutely continuous with respect to the product measure of Lebesgue measure on $M_N(\mathbb{C}^d)_{\mathrm{sa}}$ and Haar measure on $U_N(\mathbb{C})$. Then $^{\exists}N_0 \in \mathbb{N}$ s.t. we have almost surely

$$(X^N, U^N) \in \mathrm{dom}(r), \quad N \geq N_0$$

Main result 2: Convergence in distribution

- $T \in \widetilde{\mathcal{M}}$: self-adjoint.
- $P_T(B)$: spectral projection on B for a Borel set B.
- μ_T: spectral measure $\mu_T(B) := \tau[P_T(B)]$
- cumulative distribution function $\mathcal{F}_T(t) = \mu_T(-\infty, t]$, $t \in \mathbb{R}$.

Theorem (Collins-Mai-M-Parraud-Yin'22)

Let r be a nc rational function. For each $N \in \mathbb{N}$, let $X^N = (X_1^N, \ldots, X_d^N)$ be a tuple of (possibly unbounded) operators affiliated with a W^*-probability space $(\mathcal{M}_N, \tau_N)$. We suppose that X^N converges in $*$-distribution towards bounded operators $X = (X_1, \ldots, X_d)$ in a W^*-probability space $(\mathcal{M}, \tau)$. We also assume $X^N, X \in \mathrm{dom}(r)$ and $r(X^N), r(X)$ are self-adjoint. Then we have for any continuous point $t \in \mathbb{R}$ of $\mathcal{F}_{r(X)}$,

$$\lim_{N\to\infty} \mathcal{F}_{r(X^N)}(t) = \mathcal{F}_{r(X)}(t).$$

Corollary of main results

By combining our results and the result in Mai-Speicher-Yin, we have the following.

Corollary

Let $(X^N, U^N) \in M_N(\mathbb{C})_{\mathrm{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2}$ be a tuple of independent GUE and Haar unitary matrices and $(X, U) \in \mathcal{M}_{\mathrm{sa}}^{d_1} \times \mathcal{U}(\mathcal{M})^{d_2}$ be a tuple of free semicircles and Haar unitaries. Then for any nc rational function r with $d_1 + d_2$ indeterminates such that $r(X, U)$ self-adjoint, the empirical eigenvalue distribution of $r(X^N, U^N)$ almost surely converges in distribution towards the spectral measure of $r(X, U)$.

Strategy of the proof

Strategy of the proof: Linearization

Proposition (Linearization)

For a nc rational expression r we can find A, u, v s.t.

$$r(X) = {}^t u A(X)^{-1} v,\ X \in \mathrm{dom}(r)\ (\text{linearization}),$$

where

- $A \in M_k(\mathbb{C}\langle x_1, \ldots, x_d \rangle)$: *linear, i.e.*

$$A = A_0 + A_1 x_1 + \cdots + A_d x_d,\ A_i \in M_k(\mathbb{C}).$$

- $u, v \in \mathbb{C}^k$.
- $\mathrm{dom}(r) \subset \{X \in \widetilde{\mathcal{M}}^d; {}^\exists A(X)^{-1}\}$.

- $\mathrm{dom}(r) \neq \emptyset \Rightarrow A$ is full, i.e. there is no $l < k$ s.t. $A = BC, B \in M_{k \times l}(\mathbb{C}\langle x_1, \ldots, x_d \rangle), C \in M_{l \times k}(\mathbb{C}\langle x_1, \ldots, x_d \rangle)$.

Algorithm for linearization

For $r_1 = {}^t u_1 A_1^{-1} v_1, \; r_2 = {}^t u_2 A_2^{-1} v_2,$

$$\begin{aligned}
r_1 + r_2 &= \begin{pmatrix} {}^t u_1 & {}^t u_2 \end{pmatrix} \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
r_1 r_2 &= \begin{pmatrix} {}^t u_1 & 0 \end{pmatrix} \left(\begin{array}{c|c} A_1 & -v_1 {}^t u_2 \\ \hline 0 & A_2 \end{array}\right)^{-1} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \\
r_1^{-1} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\begin{array}{c|c} 0 & {}^t u_1 \\ \hline v_1 & A_1 \end{array}\right)^{-1} \begin{pmatrix} -1 \\ 0 \end{pmatrix}.
\end{aligned}$$

Algorithm for linearization

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r_1 r_2 &= \begin{pmatrix} {}^t u_1 & 0 \end{pmatrix} \left(\begin{array}{c|c} A_1 & -v_1 {}^t u_2 \\ \hline 0 & A_2 \end{array}\right)^{-1} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \\
r_1^{-1} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\begin{array}{c|c} 0 & {}^t u_1 \\ \hline v_1 & A_1 \end{array}\right)^{-1} \begin{pmatrix} -1 \\ 0 \end{pmatrix}.
\end{aligned}$$

In the third equality, one can see from the formal calculation,

$$\left(\begin{array}{c|c} 0 & {}^t u \\ \hline v & A \end{array}\right)^{-1} = \left(\begin{array}{c|c} -r^{-1} & r^{-1} {}^t u A^{-1} \\ \hline A^{-1} v r^{-1} & A^{-1} - A^{-1} v r^{-1} {}^t u A^{-1} \end{array}\right).$$

Examples of linearization

$$x_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_1 x_2 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{cc|cc} 1 & -x_1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 1 & -x_2 \\ 0 & 0 & 0 & 1 \end{array}\right)^{-1} \left(\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 1 \end{array}\right)$$

$$(x_1 x_2)^{-1} = \left(\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c|cccc} 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & -x_1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -x_2 \\ 1 & 0 & 0 & 0 & 1 \end{array}\right)^{-1} \left(\begin{array}{c} -1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right)$$

Self-adjoint linearization

Theorem (J.W.Helton, T.Mai and R.Speicher '18)

Let r be a rational expression and $\mathcal{A}$ be a $*$-algebra. If $r(X)$ is self-adjoint for $X \in \mathcal{A}^d$, then there exists $A \in M_k(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ and $u \in \mathbb{C}^k$ s.t.

$$A = \sum_{i=1}^{d} A_i \otimes x_i, \quad A_i^* = A_i.$$

$$r(X) = u^* A^{-1}(X) u.$$

For $r = {}^t u A^{-1} v$, consider

$$\begin{pmatrix} v^* & {}^t u \end{pmatrix} \begin{pmatrix} 0 & A(X) \\ A^*(X) & 0 \end{pmatrix}^{-1} \begin{pmatrix} v \\ \overline{u} \end{pmatrix},$$

which represents $2r(X)$ since $r(X)$ is self-adjoint.

Remark about main result 1

- For the proof, we use ${}^{\forall}r$, $\mathrm{dom}(r) \cap M_N(\mathbb{C})^d \neq \emptyset$ for $N \geq {}^{\exists}N_0$.
- Consider a linearization $r = {}^t u A^{-1} v$, and check by using complex analysis technique,

$$\begin{aligned} \mathrm{dom}(A^{-1}) &\cap M_N(\mathbb{C})_{\mathrm{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2} = \emptyset \\ &\Longrightarrow \mathrm{dom}(A^{-1}) \cap M_N(\mathbb{C})^{d_1+d_2} = \emptyset. \end{aligned}$$

- If X_1, X_2 are $N \times N$ symmetric matrices, then

$$\begin{aligned} \det(X_1X_2 - X_2X_1) &= \det({}^t(X_1X_2 - X_2X_1)) \\ &= (-1)^N \det(X_1X_2 - X_2X_1) \\ &= 0 \quad (N \text{ is odd}) \end{aligned}$$

- This imples $\mathrm{Sym}_N(\mathbb{C}) \not\subset \mathrm{dom}((x_1x_2 - x_2x_1)^{-1})$ for odd N.

Strategy for the Main result 2: estimation of the cumulative distribution function

Estimation of the cumulative distribution function

- $\mathrm{rank}(T) := \tau(P_T)$, P_T: orthogonal projection onto $\overline{\mathrm{Im}\,T}$

Lemma (Bercovici-Voiculescu'93)

For any $t \in \mathbb{R}$, we have

$$\mathcal{F}_T(t) = \max\{\tau(p);\ p \in \mathcal{P}(\mathcal{M}),\ tp \geq pTp\}.$$

Lemma

For $X, Y \in \widetilde{\mathcal{M}}_{\mathrm{sa}}$, we have

$$\sup_{t \in \mathbb{R}} |\mathcal{F}_{X+Y}(t) - \mathcal{F}_X(t)| \leq \mathrm{rank}(Y).$$

Strategy for the main result 2: Truncation

- Let $\epsilon > 0$. Approximate the function $g : x \to x^{-1}$ by continuous functions f_ϵ.
- We take f_ϵ as a continuous function such that $f_\epsilon = g$ on $\mathbb{R} \setminus [-\epsilon, \epsilon]$.
- Let $r = w^* Q^{-1} w$ be a self-adjoint linearization. We put $Q_N = Q(X^N)$, $Q_\infty = Q(X)$. Then

$$\begin{aligned} |\mathcal{F}_{w^* Q_N^{-1} w}(t) - \mathcal{F}_{w^* Q_\infty^{-1} w}(t)| \;\; \leq \;\; & |\mathcal{F}_{w^* Q_N^{-1} w}(t) - \mathcal{F}_{w^* f_\epsilon(Q_N) w}(t)| \\ & + |\mathcal{F}_{w^* f_\epsilon(Q_N) w}(t) - \mathcal{F}_{w^* f_\epsilon(Q_\infty) w}(t)| \\ & + |\mathcal{F}_{w^* f_\epsilon(Q_\infty) w}(t) - \mathcal{F}_{w^* Q_\infty^{-1} w}(t)|. \end{aligned}$$

Strategy for the main result 2: Rank estimation

- From previous Lemma, we have for $X = Q_N, Q_\infty$ ($k \times k$ operator valued matrices),

$$
\begin{aligned}
|\mathcal{F}_{w^* X^{-1} w}(t) - \mathcal{F}_{w^* f_\epsilon(X) w}(t)| &\leq \mathrm{rank}(w^*(X^{-1} - f_\epsilon(X))w) \\
&\leq k \times \mathrm{rank}(ww^*(X^{-1} - f_\epsilon(X))ww^*) \\
&\leq k \times \mathrm{rank}(X^{-1} - f_\epsilon(X)) \\
&\leq \mathrm{Tr}_k \otimes \tau(1_{[-\epsilon,\epsilon]}(X)).
\end{aligned}
$$

- $\lim_{\epsilon \to 0} \mathrm{Tr}_k \otimes \tau(1_{[-\epsilon,\epsilon]}(Q_\infty)) = \mathrm{Tr}_k \otimes \tau(1_{\{0\}}(Q_\infty)) = 0$ since Q_∞ is invertible.

Strategy for the main result 2: Norm estimation

- For $|\mathcal{F}_{w^* f_\epsilon(Q_N) w}(t) - \mathcal{F}_{w^* f_\epsilon(Q_\infty) w}(t)|$, we show the convergence in moments

$$\limsup_{N\to\infty} |\tau_N[(w^* f_\epsilon(Q_N) w)^l] - \tau[(w^* f_\epsilon(Q_\infty) w)^l]| = 0$$

- We use the assumption Q_∞ is bounded, and we approximate f_ϵ by a polynomial P on $[-\|Q_\infty\| - 1, \|Q_\infty\| + 1]$.

- For $|\tau_N[(w^* P(Q_N) w)^l] - \tau[(w^* P(Q_\infty) w)^l]|$, we can use the assumption of convergence in $*$-joint moments.

- For $|\tau_N[(w^* f_\epsilon(Q_N) w)^l] - \tau_N[(w^* P(Q_N) w)^l]|$, we need additional estimate.

Future perspective

- Positivity of nc rational functions evaluated in free random variables (Cf. Helton'02)
- Other analytic properties of $\mu_{r(X)}$ (e.g. absolute continuity)
- The case where some variables are commuting (normal operators, ϵ-free, bi-free).

Future perspective

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Thank you for your attention!