Probabilistic Operator Algebra Seminar

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Title: *Commutator estimates in W^*-algebras and applications*

Let $\mathcal{M}$ be a W^*-algebra and let $Z(\mathcal{M})$ be the center of $\mathcal{M}$. Fix an element $a \in \mathcal{M}$ and set $\delta_a(\cdot) := [a, \cdot]$. Obviously, δ_a is a linear bounded operator on $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$, where $\|\cdot\|_{\mathcal{M}}$ is a C^*-norm on $\mathcal{M}$. It is well known (see e.g. [3, Theorem 4.1.6]) that there exists $c \in Z(\mathcal{M})$ such that the following estimate holds: $\|\delta_a\| \geq \|a - c\|_{\mathcal{M}}$. In view of this result, it is natural to ask whether there exists an element $y \in \mathcal{M}$ with $\|y\| \leq 1$ and $c \in Z(\mathcal{M})$ such that $|[a, y]| \geq |a - c|$?

The following estimate easily follows from our main result (joint work with A. Ber (Tashkent)): for every self-adjoint element $a \in \mathcal{M}$ there exists an element $c \in Z(\mathcal{M})$ and the family $\{u_\varepsilon\}_{\varepsilon>0}$ of unitary operators from $\mathcal{M}$ such that

$$|\delta_a(u_\varepsilon)| \geq (1 - \varepsilon)|a - c|, \ \forall \varepsilon > 0. \tag{1}$$

In the special case, when $\mathcal{M}$ is finite, or else purely infinite σ-finite, there exists a unitary operator u_0 from $\mathcal{M}$ and $c \in Z(\mathcal{M})$ such that

$$|\delta_a(u_0)| = u_0^*|a - c|u_0 + |a - c| \geq |a - c|.$$

The estimate (1) above is actually sharp and, with its aid, we shall easily show that every derivation δ on $\mathcal{M}$ taking its values in a (not necessary $\|\cdot\|_{\mathcal{M}}$-closed) two-sided ideal $I \subset \mathcal{M}$ has the form $\delta = \delta_a$, where $a \in I$.

Further restatements yield complements to classical results of J. Calkin [1] and M.J. Hoffman [2] obtained originally for the special case when $\mathcal{M}$ coincides with the algebra $B(H)$ of all bounded linear operators on a separable Hilbert space H.

Analogous results continue to hold in the setting of the theory of non-commutative integration initiated by I.E. Segal [5]. In this general setting, the W^*-algebra $\mathcal{M}$ is replaced with larger algebras of "measurable" operators affiliated with $\mathcal{M}$ and the ideal I in $\mathcal{M}$ is replaced with an $\mathcal{M}$-bimodule of measurable operators. In particular, we show that estimate (1) holds for every self-adjoint element a from the classical algebra of all locally measurable operators $LS(\mathcal{M})$ which is the most general algebra in noncommutative integration to date (see [4]) and an element c from the center of $LS(\mathcal{M})$.

We also specialize our results to 'symmetric ideals of measurable operators'.

References

[1] J. Calkin, *Two-sided ideals and congruences in the ring of bounded operators in Hilbert space,* Ann. of Math. **42** (1941), 839–873.

[2] M. J. Hoffman, *Essential commutants and multiplier ideals,* Indiana Univer. Math. J. **30** (1981), No. 6, 859–869.

[3] S. Sakai, *C^*–algebras and W^*–algebras*, Springer-Verlag, New York, 1971.

[4] S. Sankaran, *The *-algebra of unbounded operators,* J. London Math. Soc. **34** (1959) 337–344.

[5] I.E. Segal, *A non-commutative extension of abstract integration*, Ann. of Math. **57** (1953) 401–457.