Math 55 Section Worksheet

GSI: Jeremy Meza
Office Hours: Wed 10-12pm, Evans 775
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1 Warm-up

1. Suppose $a, b, c, m \in \mathbb{Z}$ and $c \mid a$ and $c \mid b$.
   True or False: If $a \equiv b \pmod{m}$, then $\frac{a}{c} \equiv \frac{b}{c} \pmod{m}$.
2. Calculate $4^{-1} \pmod{9}$. Here, 4^{-1} denotes the inverse of 4, modulo 9.
3. State Fermat's Little Theorem.

2 Problems

1. Solve the congruence $34x \equiv 77 \pmod{89}$.
2. Use Fermat's Little Theorem to find $7^{121} \pmod{13}$.
3. Show with the help of Fermat's Little Theorem that if n is a positive integer, then 42 divides $n^7 - n$ (Hint: Show that 2, 3, 7 divide it).
4. Use the Chinese Remainder Theorem to find all solutions to the system of congruences $x \equiv 2 \pmod{3}, x \equiv 1 \pmod{4}, x \equiv 3 \pmod{5}$.
5. Prove that for all $n \in \mathbb{N}$,
   $$\sum_{i=1}^{n} (-1)^i i^2 = (-1)^n \frac{n(n+1)}{2}$$
6. Define a sequence recursively by $a_1 = a_2 = 1$ and $a_n = a_{n-1} + 2a_{n-2}$. Prove that for every $n \in \mathbb{N} - \{0\}$, a_{3n} is divisible by 3.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(xy) = xf(y) + yf(x)$. Prove that for all $u \in \mathbb{R} - \{0\}$ and all $n \in \mathbb{N} - \{1\}$, $f(u^n) = nu^{n-1} f(u)$.
8. Consider the Fibonacci sequence f_n defined by setting $f_0 = 0, f_1 = 1$ and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. Prove the following:
   (a) $\sum_{i=0}^{n} f_i = f_{n+2} - 1$
   (b) $\sum_{i=0}^{n} f_i^2 = f_n \cdot f_{n+1}$
   (c) $f_{n-1} \cdot f_{n+1} - f_n^2 = (-1)^n$

3 A Fun Problem

The following text has been plagiarized verbatim from the original short story by Ben Ames Williams (1926).

The little man still hesitated, said at last reluctantly, "Why, I had it at noon from a man uptown. Haven't been able to get it yet." Marr made an impatient, peremptory gesture; and Wadlin said vaguely, "It's just a problem in indeterminates, I think."

"What is it, man?" Marr cried. "Let me in on it. I'll straighten you out in no time."

"I don't want to bother you," Wadlin argued. "It's not as simple as it looks. I've been at it six hours and more."

"What is it? What is it?" Marr demanded. "You act as though it were confidential."

So at last Wadlin told him. "Well," he explained, "according to the way the thing was given to me, five men and a monkey were shipwrecked on a desert island, and they spent the first day gathering coconuts for food. Piled them all up together and then went to sleep for the night. "But when they were all asleep one man woke up, and he thought there might be a row about dividing the coconuts in the morning, so he decided to take his share. So he divided the coconuts into five piles. He had one coconut left over, and he gave that to the monkey, and he hid his pile and put the rest all back together."

He looked at Marr; the man was listening attentively. "So by and by the next man woke up and did the same thing," Wadlin continued. "And he had one left over, and he gave it to the monkey. And all five of the men did the same thing, one after the other; each one taking a fifth of the coconuts in the pile when he woke up, and each one having one left over for the monkey. And in the morning they divided what coconuts were left, and they came out in five equal shares." He added morosely, "Of course each one must have known there were coconuts missing; but each one was guilty as the others, so they didn't say anything."

Marr asked sharply, "But what's the question?"

"How many coconuts were there in the beginning?" Wadlin meekly explained.

Marr laughed. "Why, that's simple enough. You just —"

But their victuals were served; the table was filled with viands; they could find no room for calculations. So when they were done Marr said, "Here, you come upstairs to the [Math 55 Discussion Section] and we'll work this out. It won't take long."

What is the minimum possible number of coconuts in the beginning? (Hint: Let $x - 4$ denote the number of initial coconuts.)