Math 55 Section Worksheet

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1 Welcome Back

Question: How do you solve a recurrence relation? Think of some ways you have gone over in lecture.

Question: What is *Inclusion-Exclusion?* Explain it in words and then try to write down the statement. Look at your notes and recall how you counted the number of *derangements* of $[n]$.

2 Together

Using generating functions, we will solve the recurrence relation $a_k = 5a_{k-1} - 6a_{k-2}$ with initial conditions $a_0 = 6$ and $a_1 = 30$. Before we do, let's review generating functions:

Find a closed form for the generating function for the sequences $\{a_n\}$ given below

1. $a_n = 5$ for $n \geq 0$
2. $a_n = 5$ for $n \geq 3$ and $a_0 = a_1 = a_2 = 0$.
3. $a_n = 5^n$ for $n \geq 0$.
4. $a_n = \binom{5}{n}$ for $n \geq 0$.
5. $a_n = 5n + 3$ for $n \geq 0$.

3 You Try!

1. Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$.
2. Count the number of surjective functions $f : [5] \to [3]$.
3. Find the number of natural numbers from 1 to 1000 that are neither perfect squares, cubes, nor fourth powers.
4. Suppose that p and q are distinct primes. Use the principle of inclusion-exclusion to find $\varphi(pq)$, the number of positive integers not exceeding pq that are relatively prime to pq. Can you generalize this formula to $\varphi(n)$ where $n = p_1^{a_1} \cdots p_t^{a_t}$?

5. Let $A = \{1\}$. Let's count the number of elements of $A \setminus A = \varnothing$ in a rather convoluted way. For $1 \le i \le n$, let $A_i = \{1\}$ and note then that $A = A_1 \cup \ldots \cup A_n$. Thus, $0 = |\varnothing| = |A| - |A| = 1 - |A_1 \cup \ldots \cup A_n|$. Deduce that

$$0 = 1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} - \ldots + (-1)^n \binom{n}{n}$$

Can you prove this identity in any other ways? Binomial Theorem? Combinatorially? Induction?

6. Use generating functions to prove the following identities:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$$

(Hint: look at

$(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$ and $(1+x)^{m+n} = (1+x)^m(1+x)^n$

respectively.)

4 Challenge

7. Find a generating function for the number of ways to make n dollars with denominations of \$1, \$2, \$5, and \$10 bills. (Hint: Do it for just \$1, then just \$2, and so forth. Now what operation should you do?)

8. The *Catalan numbers* C_n are given by the recurrence $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ with initial conditions $C_0 = C_1 = 1$. (see Challenge problems on last weeks worksheet).

   (a) If $C(x)$ is the generating function for C_n, show that $xC(x)^2 - C(x) + 1 = 0$.

   (b) Using the initial conditions, show that $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.

   (c) For any real number x, define $\binom{x}{n} := \frac{x(x-1)\cdots(x-n+1)}{n(n-1)\cdots 1}$. Show that if n is a positive integer, then

$$\binom{-1/2}{n} = \frac{\binom{2n}{n}}{(-4)^n}$$

   (d) Use the preceding exercises and the binomial theorem to show that

$$C(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n$$

   and hence $C_n = \frac{1}{n+1}\binom{2n}{n}$.