Math 55 Section Worksheet

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1 Warm-up

1. What does it mean for a function to be well-defined?
2. What does it mean for a function to be injective (one-to-one)?
3. What does it mean for a function to be surjective (onto)?
4. What is the definition of countable?

2 Problems

1. Let $f : A \to B$ be a function. Let $S, T \subseteq A$. For each of the following, prove it must hold or provide a counterexample.
   (a) $f(S \cap T) \subseteq f(S) \cap f(T)$
   (b) $f(S \cap T) \supseteq f(S) \cap f(T)$
2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is an increasing function, that is suppose
$$\forall x \in \mathbb{R}(x < y \longrightarrow f(x) < f(y))$$
   Prove that f is injective. Can you think of an example where f is not surjective?
3. Give an example of a function $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ that is surjective.
4. Let $f : A \to B$ and $g : B \to C$ be functions. Suppose $g \circ f$ is injective. Prove that f is injective. If f is injective, is it necessarily true that $g \circ f$ is injective?
5. For a set S and $n \in \mathbb{N} - \{0\}$, we denote S^n as the Cartesian product of S with itself n times. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by
$$f(x, y, z) = (xy, yz)$$
   Is f injective? surjective? Prove your claims.
6. Evaluate the sum $\sum_{j=0}^{8} 3 \cdot 2^j$.
7. Let S be the set of all infinite binary strings (that is, elements of S are infinitely long strings of 0s and 1s). Provide a bijection between S and $\mathcal{P}(N)$.

3 Extra

8. Let $[n]$ denote the set $\{1, \ldots, n\}$. Let A be the set of subsets of $[n]$ that have even size and B be the set of subsets of $[n]$ that have odd size. Establish a bijection between A and B.

9. Let f, g be surjections from $\mathbb{Z}$ to $\mathbb{Z}$ and let $h = fg$ be their product. Must h also be surjective?

10. Let A, B be countable sets.

    (a) Prove that if $A \cap B = \varnothing$, then $A \cup B$ is countable.

    (b) Prove in general that $A \cup B$ is countable.

11. What do you think are the cardinalities of the following sets?

    (a) The set of all infinite binary strings.

    (b) The set of all infinite binary strings with only *finitely* many 1s.

    (c) The set of functions from $\mathbb{N}$ to $\mathbb{N}$.

    (d) The set of functions from $\mathbb{N}$ to $\{1, 2\}$.

    (e) $\bigcup_{n \in \mathbb{N}} \{0, 1, \ldots, n\}$.

    (f) $\bigcup_{x \in \mathbb{R}} \{x\}$.

    (g) The rational numbers.

    (h) The irrational numbers.

    (i) $\mathbb{N} \times \mathbb{Z} \times \mathbb{Q}$.

    (j) $\mathcal{P}(\mathbb{N})$.

    (k) $\mathcal{P}(\mathbb{R})$.

    (l) $\mathbb{C}$

    (m) A basis of $\mathbb{R}$ as a vector space over $\mathbb{Q}$.