Math 55 Worksheet 9

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1 We Induct Together

1. Prove that $2^n + 1$ is divisible by 3 for all odd integers n.

2. What is wrong with this "proof"?

   Theorem: $\frac{d}{dx}x^n = 0$ for all $n \geq 0$.

   Base Step: $n = 0$. $\frac{d}{dx}x^0 = \frac{d}{dx}1 = 0$.

   Inductive Step: Assume that $\frac{d}{dx}x^k = 0$ for all $k \leq n$ for some fixed but arbitrary $n \geq 0$. Then, by the product rule,

$$\frac{d}{dx}x^{n+1} = x^n \frac{d}{dx}x^1 + x^1 \frac{d}{dx}x^n = x^n \cdot 0 + x^1 \cdot 0 = 0$$

2 You Induct Together

1. Consider the Fibonacci sequence f_n defined by setting $f_0 = 0, f_1 = 1$ and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. Prove the following:
   (a) $\sum_{i=0}^{n} f_i = f_{n+2} - 1$
   (b) $\sum_{i=0}^{n} f_i^2 = f_n \cdot f_{n+1}$
   (c) $f_{n-1} \cdot f_{n+1} - f_n^2 = (-1)^n$

2. Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using induction.

3. Prove that a convex n-gon has $n(n-3)/2$ diagonals.

4. Consider a chessboard of size $2^n \times 2^n$ for some arbitrary natural number n. Remove any square from the board. Is it possible to tile the remaining squares with L-shaped triominoes? (An L-shaped triomino is a tile with 3 squares in the shape of an L.) If so, prove it. If not, provide a counterexample.

5. Let W be the set of *balanced strings of parentheses*, that is, an element $w \in W$ satisfies one of the following:

(i) w is the string "( )".

(ii) $\exists u \in W$ such that w is the string "(u)".

(iii) $\exists u, v \in W$ such that w is the string "$u\,v$".

For example, "(()(()))" is a balanced string of parentheses. Prove that in any string w, there is the same number of left parentheses as right parentheses. (Hints: first define terms and write what you want to prove using logical symbols. Then induct on the length of the string).

Challenge: Count the number of balanced strings of parentheses of length $2n$. You can try finding a recursive formula and proving it by induction.