Math 55 Worksheet 8

Jeremy Meza

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1 Together

1. Prove that for all $n \in \mathbb{N}$,
$$\sum_{i=1}^{n}(-1)^i i^2 = (-1)^n \frac{n(n+1)}{2}$$
2. What is wrong with this "proof"?

   Theorem: For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.

   Base Step: Suppose that $n = 1$. If $\max(x, y) = 1$ and x, y are positive integers, then $x = 1$ and $y = 1$.

   Inductive Step: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ and x, y are positive integers, then $x = y$. Now let $\max(x, y) = k + 1$ for some positive integers x, y. Then, $\max(x - 1, y - 1) = k$, so by the inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$.

2 Induction: You Try!

1. Prove that for all $n \in \mathbb{N}$ with $n \geq 1$,
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$
2. Define a sequence recursively by $a_1 = a_2 = 1$ and $a_n = a_{n-1} + 2a_{n-2}$. Prove that for every $n \in \mathbb{Z}_+$, a_{3n} is divisible by 3.
3. Prove that $n^3 + 2n$ is divisible by 3 for all integers n.
4. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(xy) = xf(y) + yf(x)$. Prove that for all $u \in \mathbb{R} - \{0\}$ and all $n \in \mathbb{N}$, $f(u^n) = nu^{n-1}f(u)$.
5. Consider the Fibonacci sequence f_n defined by setting $f_0 = 0, f_1 = 1$ and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. Prove the following:
   (a) $\sum_{i=0}^{n} f_i = f_{n+2} - 1$
   (b) $\sum_{i=0}^{n} f_i^2 = f_n \cdot f_{n+1}$
   (c) $f_{n-1} \cdot f_{n+1} - f_n^2 = (-1)^n$