Math 55 Midterm Review

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1 Warm-up

- Turn to your neighbors and come up with topics you would like reviewed.
- Read the handout "How to lose marks on math exams".
- Ask a question!

2 Definitions

Give precise mathematical definitions of the following terms. Give counterexamples where necessary so that you truly understand what the term means. How would you start a proof in which you need to show that something satisfies the term?

(a) Subset

(b) Injection

(c) Surjection

(d) Bijection

(e) Countable

(f) gcd

(g) Congruence modulo m

3 Proofs

1. Find a set S such that $\mathcal{P}(S) = \varnothing$ or explain why no such set exists.

2. Let $a, b, c \in \mathbb{Z}$ such that $a^2 + b^2 = c^2$. Prove that at least one of a, b is even. (Hint: Look at $c^2 \bmod 4$).

3. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by $f(x, y) = 2^{x-1}(2y - 1)$. Prove that f is injective.

4. Prove that if p is prime, the only solutions of $x^2 \equiv 1 \pmod{p}$ are integers x such that $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$.

5. Let $f : A \to B$. Recall that for $S \subseteq B$, we define $f^{-1}(S) := \{x \in A \mid f(x) \in S\}$. Prove or provide a counterexample that $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$.

6. Show with the help of Fermat's Little Theorem that if n is a positive integer, then 42 divides $n^7 - n$ (Hint: Show that 2, 3, 7 divide it).

7. Prove that if m is a positive integer of the form $4k+3$ for some non-negative integer k, then m is not the sum of the squares of two integers.

4 Computation

8. Let A, B be sets and P, Q, R propositions. Negate the following proposition:
$$\exists x \in A \; \forall y \in B \; \forall z \in A, \; P(x, y, z) \to (Q(z) \wedge R(x))$$

9. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x - 3$. Let $g : \mathbb{Z} \to \mathbb{N}$ be defined by $g(x) = |x| + 4$. What is the domain of $g \circ f$? What is the codomain? Is this function injective? Surjective?

10. Let $\mathbb{R}$ be the universe and $P(x, y)$ be the statement $x + y \in \mathbb{Q}$. Determine the truth values of the following statements:
    (a) $\forall x \in \overline{\mathbb{Q}} \; \exists y \in \overline{\mathbb{Q}} \; P(x, y)$
    (b) $\exists x \in \overline{\mathbb{Q}} \; \forall y \in \overline{\mathbb{Q}} \; P(x, y)$

11. Evaluate $(3^{99} + 6^4) \bmod 8$.

12. Solve the congruence $34x \equiv 77 \pmod{89}$.

13. Is $\{2, \varnothing\} \subseteq \mathcal{P}(\mathbb{Z} \cup \{\varnothing\})$?

14. Use Fermat's Little Theorem to find $7^{121} \pmod{13}$.

15. Use the Chinese Remainder Theorem to find all solutions to the system of congruences $x \equiv 2 \pmod{3}, x \equiv 1 \pmod{4}, x \equiv 3 \pmod{5}$.