Math 55 Worksheet 5

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OH: Tues 10-12pm, Evans 1047

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1 Warm-up

1. What does it mean for a function to be injective (one-to-one)?
2. What does it mean for a function to be surjective (onto)?
3. What is the definition of countable?

2 Problems

1. Let $f : A \to B$ be a function. Given a subset $S \subseteq A$, we define the *image of S under f* to be the set

   $$f(S) := \{y \in B \mid \exists x \in S, f(x) = y\}$$

   Let $S, T \subseteq A$. For each of the following, prove it must hold or provide a counterexample.

   (a) $f(S \cap T) \subseteq f(S) \cap f(T)$

   (b) $f(S \cap T) \supseteq f(S) \cap f(T)$

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is an increasing function, that is suppose

   $$\forall x \in \mathbb{R}(x < y \longrightarrow f(x) < f(y))$$

   Prove that f is injective. Can you think of an example where f is not surjective?

3. Give an example of a function $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ that is surjective.

4. Let S be the set of all infinite binary strings (that is, elements of S are infinitely long strings of 0s and 1s). Provide a bijection between S and $\mathcal{P}(\mathbb{N})$.

5. Let $[n]$ denote the set $\{1, \ldots, n\}$. Let A be the set of subsets of $[n]$ that have even size and B be the set of subsets of $[n]$ that have odd size. Establish a bijection between A and B.

6. (**hard**) Let A, B be countable sets.

   (a) Prove that if $A \cap B = \varnothing$, then $A \cup B$ is countable.

   (b) Prove in general that $A \cup B$ is countable.

3 Extra

7. What do you think are the cardinalities of the following sets?

   (a) The set of all infinite binary strings.

   (b) The set of all infinite binary strings with only finitely many 1s.

   (c) The set of functions from $\mathbb{N}$ to $\mathbb{N}$.

   (d) The set of functions from $\mathbb{N}$ to $\{1,2\}$.

   (e) $\bigcup_{n \in \mathbb{N}} [n]$.

   (f) $\bigcup_{x \in \mathbb{R}} \{x\}$.

   (g) The rational numbers.

   (h) The irrational numbers.

   (i) $\mathbb{N} \times \mathbb{Z} \times \mathbb{Q}$.

   (j) $\mathcal{P}(\mathbb{N})$.

   (k) $\mathcal{P}(\mathbb{R})$.

   (l) $\mathcal{P}(\mathcal{P}(\mathbb{N}))$.

   (m) The number of different cardinalities.

   (n) The number of bijections from $[n]$ to itself.

   (o) $\mathbb{C}$

   (p) Any set that contains an uncountable set.

   (q) Any set contained in a countable set.

   (r) Any set that has a surjection onto an uncountable set.

   (s) Any set that has an injection into a countable set.

   (t) All the grains of sand in the world.

   (u) All the quarks in all the protons in all the nuclei in all the atoms in all the particles in all of the universe.

   (v) A basis of $\mathbb{R}$ as a vector space over $\mathbb{Q}$.

   (w) The set of all polynomials with integer coefficients.

   (x) The set of algebraic numbers.

   (y) The set of transcendental numbers.

   (z) Your patience with this problem.