Math 55 Worksheet 4

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OH: Tues 10-12pm, Evans 1047

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1 Warm-up

1. What are the sets $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}_+, \mathbb{R} - \mathbb{Q}$?
2. What is another name the following set is known by: $\{x \in \mathbb{R} \mid \frac{x}{2} \in \mathbb{Z}\}$.
3. Give sets S, T, what do $\overline{S}, S \cap T$, $S \cup T$, and $S - T$ denote?
4. Given a set S, what is $\mathcal{P}(S)$?
5. True or False?
   (a) $\{1\} \in \{1\}$.
   (b) $\{1\} \subseteq \{1\}$.
   (c) $\varnothing \subseteq \{1\}$.
   (d) $\varnothing \in \{1\}$.
   (e) $\varnothing \in \mathcal{P}(\{1\})$
   (f) $\{\varnothing\} \subseteq \{\varnothing, \{\varnothing\}\}$.
   (g) $\{\{1\}, 2\} \subseteq \{1, 2, \{1, 2\}\}$.
   (h) There are 8 elements in $\mathcal{P}(\{1, \{1, 2\}, \varnothing\})$.
6. Let $f : A \to B$ be a function between two sets A, B. What is the domain of f? What is the codomain? The image? The range?
7. What does it mean for a function to be injective (one-to-one)?
8. What does it mean for a function to be surjective (onto)?

2 Problems

1. Prove that $A = A \cap B$ if and only if $A \subseteq B$.
2. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

3. Let $f : A \to B$ be a function. Let $S, T \subseteq A$. For each of the following, prove it must hold or provide a counterexample.

   (a) $f(S \cap T) \subseteq f(S) \cap f(T)$

   (b) $f(S \cap T) \supseteq f(S) \cap f(T)$

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is an increasing function, that is suppose

$$\forall x \in \mathbb{R}(x < y \longrightarrow f(x) < f(y))$$

   Prove that f is injective. Can you think of an example where f is not surjective?

5. Give an example of a function $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ that is surjective.

6. Determine whether each of the following are injections, surjections, bijections, or none of the three.

   (a) $f : \mathbb{R} \to \mathbb{R}, f(x) = 3x - 2$

   (b) $f : \mathbb{R} \to \mathbb{Z}, f(x) = \lfloor x \rfloor$

   (c) $f : \mathbb{N} \to \mathbb{N}, f(n) = n + 1$

   (d) $f : \mathbb{R} \to \mathbb{R}, f(x) = x(x - 3)(x + 2)$

7. For a set S and $n \in \mathbb{N} - \{0\}$, we denote S^n as the Cartesian product of S with itself n times. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f(x, y, z) = (xy, yz)$$

   Is f injective? surjective? Prove your claims.

8. Let $f : A \to B$ and $g : B \to C$ be functions. Suppose $g \circ f$ is injective. Prove that f is injective. If f is injective, is it necessarily true that $g \circ f$ is injective?