Math 55 Worksheet 16

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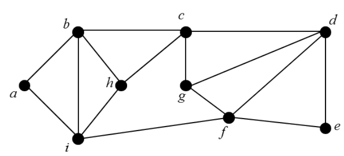
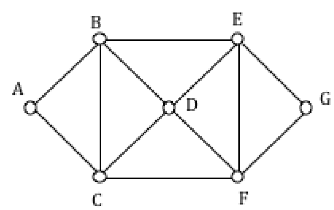
Monday, December 2

1 Together

1. What is an *Euler circuit*? Find an Euler circuit in the following graphs or explain why one does not exist.





2. Consider the following

Theorem 1.1. *A connected graph G with at least 2 vertices has an Euler circuit iff each vertex has even degree.*

How does this proof work? If G has an Euler circuit, why is it necessary for each vertex to have even degree? Conversely, if every vertex of G has even degree, how do we construct an Euler circuit?

2 Problems

1. A graph is *planar* if it can be drawn without any edges intersecting (edges don't have to be straight lines). Which of the following graphs do you think are planar?

   (a) K_3? K_4? K_5? (b) $K_{2,2}$? $K_{3,3}$?

2. A *tree* is a connected graph without cycles. Show that the following are equivalent definitions of a tree:

   (a) A maximally acyclic graph (i.e. adding any edge will result in a cycle)

   (b) A minimally connected graph (i.e. removing any edge will result in a disconnected graph).

(c) A graph such that there exists a unique path between any two vertices.

3 Challenges

3. Solve the problems that Matt Damon solved in *Good Will Hunting*:

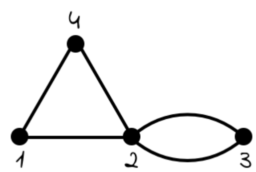
   (a) Given the graph G below, find

      i. The adjacency matrix A

      ii. The matrix giving the number of 3 step walks

      iii. The generating function for walks from $i \to j$

      iv. The generating function for walks from $1 \to 3$.



   (b) How many (non-isomorphic) homeomorphically irreducible trees are there on 10 vertices? A tree is *homeomorphically irreducible* if it has no vertices of degree 2.

4. A binary tree is a tree such that every vertex has either 0 or 2 children. A leaf of a tree is a vertex with no children. Count the number of binary trees with $n = 1, 2, 3, 4, 5, 6$ leaves. Can you find a recursive formula?

5. The graph $K_{3,3}$ is not planar, that is, it cannot be *embedded in the plane.* However, it can be *embedded in a torus.* You can visualize a torus as a square paper with opposite sides identified, so that an edge that goes out the right side comes in on the left side, and an edge that goes out the bottom side comes out on the top side. Can you draw $K_{3,3}$ on this square paper with no crossing edges?

6. A *Hamilton path* is a path that passes through every vertex exactly once. A *Hamilton cycle* is a Hamilton path that starts and ends at the same vertex. A graph is *Hamiltonian* if it has a Hamilton cycle.

   Let's play a game. This game will involve two players: Player 1 and Player 2. We start with an empty graph on N vertices. Players take turns drawing any edge between 2 vertices under the condition that no vertex can have degree exceeding 4 (no multiple edges or loops are allowed). Play continues until no more edges can be added to the graph. Player 1 wins if the resulting graph is Hamiltonian and Player 2 wins if the resulting graph is not Hamiltonian.

   Find someone in the class and try playing this game with $N = 5, 6, 7, 8, \dots$ vertices. Which player do you think has a winning strategy?