Math 55 Worksheet 13

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1 Together

Turn to a neighbor and answer the following questions:

(a) What is a *linear homogenous recurrence relation of degree* k?

(b) What is a *characteristic equation*? How can we use it to find a solution to a recurrence relation?

(c) What is a *generating function*?

Let's solve the recurrence relation $a_k = 5a_{k-1} - 6a_{k-2}$ with initial conditions $a_0 = 6$ and $a_1 = 30$. Let's first use the method of characteristic equations:

Now let's use generating functions. Let's first warm up by finding a closed form for the generating functions of the sequences $\{a_n\}$ given below

1. $a_n = 5$ for $n \geq 0$
2. $a_n = 5$ for $n \geq 3$ and $a_0 = a_1 = a_2 = 0$.
3. $a_n = 5^n$ for $n \geq 0$.
4. $a_n = \binom{5}{n}$ for $n \geq 0$.
5. $a_n = 5n + 3$ for $n \geq 0$.

Now back to the original recurrence relation:

2 Problems

1. (a) Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0's.

   (b) What are the initial conditions?

   (c) How many bit strings of length 5 contain two consecutive 0s?

2. Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 6$, $a_1 = 8$.

3. In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?

4. Let $S(n,k)$ denote the number of ways to partition the set $\{1, \ldots, n\}$ into k nonempty subsets. For example, $S(3,2) = 3$ because we have the partitions $\{\{1\},\{2,3\}\}, \{\{2\},\{1,3\}\}$, and $\{\{3\},\{1,2\}\}$. We set the initial conditions $S(0,0) = 1$ and $S(n,0) = S(0,n) = 0$ for $n > 0$. Find a recurrence relation for $S(n,k)$.

5. Let p_n be the number of ways to group $2n$ distinct people into pairs.

   (a) Find a recursive formula for p_n.

   (b) Give a combinatorial argument to show that $p_n = \frac{(2n)!}{n!2^n}$.

6. Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$.

7. Use generating functions to prove the following identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

   (Hint: multiply by x^k and take a sum.)

3 Challenge

1. A *triangulation* of a convex polygon is a decomposition of the polygon into triangles by connecting vertices with non-crossing line segments. Below are the triangulations of a hexagon:

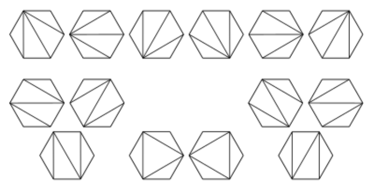

Figure 1: $C_4 = 14$ triangulations of a hexagon.

   (a) Let $C_0 = 1$ and for $n \geq 1$, let C_n be the number of triangulations of an $(n+2)$-gon. Show that C_n is given by the following recursive formula:
$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

   (b) Let $C'_0 = 1$ and for $n \geq 1$, let C'_n be the number of lattice paths to (n,n) that do not go below the main diagonal $y = x$. Show that $C'_n = C_n$ by showing that C'_n satisfies the same recurrence relation above. (Hint: case on when the lattice path first touches the diagonal).

2. The *Catalan numbers* C_n are given by the recurrence $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ with initial conditions $C_0 = C_1 = 1$.

   (a) If $C(x)$ is the generating function for C_n, show that $xC(x)^2 - C(x) + 1 = 0$.

   (b) Using the initial conditions, show that $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.

   (c) For any real number x, define $\binom{x}{n} := \frac{x(x-1)\cdots(x-n+1)}{n(n-1)\cdots 1}$. Show that if n is a positive integer, then
$$\binom{-1/2}{n} = \frac{\binom{2n}{n}}{(-4)^n}$$

   (d) Use the preceding exercises and the binomial theorem to show that
$$C(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n$$
and hence $C_n = \frac{1}{n+1}\binom{2n}{n}$.

3. *(If you know linear algebra).* Consider the Fibonacci sequence F_n. We can write this as

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} \qquad \text{with} \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

   (a) Prove that $\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$.

   (b) Calculate A^{n-1} by diagonalizing A.

   (c) Derive Binet's formula for F_n:

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

4. Recall that $S(n,k)$ denotes the number of ways to block n elements into k nonempty subsets.

   (a) Use $S(n,k)$ to count the number of surjective functions $f : [n] \to [k]$.

   (b) Let $(x)_k$ denote the *falling factorial* $(x)_k := x(x-1)(x-2)\cdots(x-k+1)$. By counting the total number of functions $f : [n] \to [x]$ in 2 ways, deduce the equality

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k$$