Math 55 Worksheet 12

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1 Introduction

The first thing to note about probability theory is that it is often **tricky** and **counterintuitive**. I encourage you to ask questions! The first occurrence of when our intuition fails us is in the celebrated **Monty Hall problem**.

Problem 1: Find someone in this class. Explain to them the Monty Hall problem and make sure you both understand why it's advantageous to switch doors when prompted.

Problem 2: Fill in the formula below for Bayes' rule. Turn to someone in the class and explain what conditional probability is, perhaps using a Venn diagram to help you visualize.

Theorem 1.1 (Bayes' Theorem)**.** *Suppose that A and B are events from a sample space S. Then,*

$$P(A \mid B) =$$

Example 1.1. Suppose that E and F are events in a sample space and $p(E) = 1/3, p(F) = 1/2$, and $p(E|F) = 2/5$. Find $p(F \mid E)$.

Problem 3: Complete the definition: Events A, B are *independent* if ______________________________.

Example 1.2. Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Are E and F independent?

Problem 4: Turn to a neighbor and try to fill in the following:
A *random variable* X on a sample space S is ____.
The *expected value* $E[X]$ of a random variable X is given by

$$E[X] =$$

The *variance* $V(X)$ of a random variable X is given by

$$V(X) = \qquad\qquad\qquad\qquad =$$

The *standard deviation* $\sigma(X)$ is related to the variance via: __________________________.

Example 1.3. What is the expected number of times a 6 appears when a fair die is rolled 10 times?

Theorem 1.2 (Linearity of Expectation)**.** *Let $X_1, \ldots, X_n$ be random variables. Then,*

$$\underline{\hspace{6cm}} = \underline{\hspace{6cm}}$$

Let's give some intuition to variance:

Theorem 1.3 (Chebyshev's Inequality)**.** *Let X be a random variable and let $r > 0$. Then,*

$$P(|X - E[X]| \geq r) \leq \underline{\hspace{5cm}}$$

2 Probabilistic Problems

Bayes' Rule

1. Suppose that one person in 10,000 people has a rare genetic disease. There is an excellent test for the disease; 99.9% of people with the disease test positive and only 0.02% who do not have the disease test positive.
   (a) What is the probability that someone who tests positive has the genetic disease?
   (b) What is the probability that someone who tests negative does not have the disease?
2. A masked container contains 2 beads, each of which is either black or red. What is the probability that both beads are black, given that you select a black bead from the container?

Random Variables

3. Suppose that you flip a fair coin n times.
   (a) What is the expected number of heads?
   (b) What is the variance of the number of heads?
4. You've just gotten your wisdom teeth pulled and have recently exited the dentist's office still delirious from anesthesia. You make it back to your apartment, pull out your keychain of n keys, but can't quite manage to figure out which key is the right one. So, you try each key one by one. What is the expected number of selections you will make until you are soundly asleep in your apartment if
   (a) you choose the keys without replacement?
   (b) you choose the keys with replacement?
5. Let X_n be the random variable that equals the number of tails minus the number of heads when n fair coins are flipped.
   (a) What is the expected value of X_n?
   (b) What is the variance of X_n?