Math 54 Section Worksheet 9 Solutions

GSI: Jeremy Meza
Office Hours: Monday 3:30-5:30pm, Evans 1047
Monday, March 2, 2020

1 Warm-Up

- Find someone next to you and define what a *basis* of a subspace is. How does a basis allow you to write a vector in a vector space as a column vector in some $\mathbb{R}^n$?

  A basis of a subspace is a linearly independent set of vectors that spans the subspace.

- One of you define what the *kernel* of a linear transformation is. How does the kernel relate to being one-to-one or onto?

  The kernel of a linear transformation is the set of all vectors in the domain that are mapped to 0 by the linear transformation. In particular, if $T : V \to W$, then $\ker T = \{v \in V \mid T(v) = 0\}$. A linear transformation is T is one-to-one if and only if $\ker T = \{0\}$.

- One of you define what the *image* of a linear transformation is. How does the image relate to being one-to-one or onto?

  If $T : V \to W$, then $\operatorname{Image} T = \{w \in W \mid w = T(v) \text{ for some } v \in V\}$. T is onto if and only if $\operatorname{Image} T = W$.

2 Problems

For the following problems, we let $\mathbb{P}_n = \{\text{polynomials with real coefficients of degree} \le n\}$.

1. Let $V = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$.

   (a) Convince yourself that V is a vector space.

   There is a 0 function, and addition and scalar multiplication of functions makes sense, keeps you in V, and satisfies the necessary distributivity, associativity and commutativity properties.

   (b) Let $W = \{f \in V \mid f(1) = 0\}$. Show that W is a subspace.

   The 0 function is in W because evaluating the 0 function at 1 is 0. If $f, g \in W$, then $(f + g)(1) = f(1) + g(1) = 0$, so $f + g$ is in W. If $c \in \mathbb{R}$, then $(cf)(1) = c(f(1)) = c(0) = 0$, so cf is in W.

2. Let $p_1(t) = 1 - 3t^2, p_2(t) = 2 + t, p_3(t) = t^3, p_4(t) = 1 + t + t^3$ be polynomials in $\mathbb{P}_3$. Determine whether p_1, p_2, p_3, p_4 are linearly independent.

Converting to the basis $\{1, t, t^2, t^3\}$, the polynomials become, respectively,

$$\begin{pmatrix} 1 \\ 0 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

After row reducing, we find a pivot in every column and so these polynomials are linearly independent.

3. Let $V = \{T : \mathbb{R}^2 \to \mathbb{R}^2 \mid T \text{ is linear}\}$ be the set of all linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

   (a) Is V a vector space? What is addition and scalar multiplication, and do they satisfy the appropriate axioms?

   Addition and scalar multiplication of linear transformations is the same as addition and scalar multiplication of functions. There is also a 0 linear transformation.

   (b) What do you think is a basis for V?

   We can represent linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ as 2×2 matrices. An arbitrary 2×2 matrix looks like $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so a convenient basis would just be the linear transformations given by the 4 matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

4. Define a linear transformation $T : \mathbb{P}_1 \to \mathbb{R}^2$ by $T(p(t)) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}$. Find polynomials that span the kernel of T and describe the image of T. Can you describe the kernel and image of T if we changed the domain to $\mathbb{P}_2$?

   We want polynomials $p(t) = a + bt$ such that $\begin{pmatrix} p(0) \\ p(1) \end{pmatrix} = 0$. So, we find that $a = 0$ and $a + b = 0$, so $p(t) = 0$. Thus, $\ker(T) = \{0\}$ and T is one-to-one. Since $\mathbb{P}_1$ and $\mathbb{R}^2$ have the same dimension, we can also conclude that T is onto and hence $\operatorname{Image} T = \mathbb{R}^2$.

   If we changed to $\mathbb{P}_2$, then we want polynomials $p(t) = a + bt + ct^2$ such that $p(0) = p(1) = 0$. We find that $a = 0$ and $a + b + c = 0$. Thus, if $p(t) \in \ker T$, then $p(t) = bt - bt^2$ for some $b \in \mathbb{R}$. In other words, $\ker T = \operatorname{Span}\{t - t^2\}$. The kernel is 1-dimensional now, with the same image as before: $\operatorname{Image} T = \mathbb{R}^2$.

5. Let $T : \mathbb{P}_2 \to \mathbb{P}_4$ be the transformation that maps a polynomial $p(t)$ to the polynomial $p(t) + t^2 p(t)$.

(a) Find the image of $p(t) = 2 - t + t^2$.
$T(2 - t + t^2) = (2 - t + t^2) + t^2(2 - t + t^2) = 2 - t + 3t^2 - t^3 + t^4$.

(b) Show that T is a linear transformation.
If $p(t), q(t) \in \mathbb{P}_2$, then

$$T(p(t) + q(t)) = (p(t) + q(t)) + t^2(p(t) + q(t))$$

We compare this to

$$T(p(t)) + T(q(t)) = (p(t) + t^2 p(t)) + (q(t) + t^2 q(t))$$

and see they are the same. A similar computation holds for $T(cp(t)) = cT(p(t))$.

(c) Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.
We compute $T(1), T(t), T(t^2)$ and write them in the basis $\{1, t, t^2, t^3, t^4\}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6. Let $V = \text{Span}\{\cos t, \sin t\}$.

(a) Convince yourself that V is a vector space. What is a basis for V?
Spans are always spaces. A basis is $\{\cos t, \sin t\}$.

(b) Define the linear transformation $D : V \to V$ by $D(f) = \frac{\partial f}{\partial t}$. In other words, $D(f)$ is the derivative of f. Can you write down a matrix for D by representing functions as column vectors?
We compute $D(\cos t), D(\sin t)$ and write them in the basis $\{\cos t, \sin t\}$:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

7. Let $I : \mathbb{P}_2 \to \mathbb{P}_3$ be the linear transformation defined by

$$I(p(t)) = \int_0^t p(x)\, dx$$

(a) Find the matrix of I with respect to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$ for $\mathbb{P}_2$ and $\mathbb{P}_3$, respectively.
We compute $T(1), T(t), T(t^2)$ and write them int he basis $\{1, t, t^2, t^3\}$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

(b) Let $D : \mathbb{P}_3 \to \mathbb{P}_2$ be the *derivative* function, defined similarly as in the previous problem. Are D and I inverses? (try computing DI and ID).

D is given by the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. We see that

$$DI = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$ID = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So, D and I are not inverses, because ID is not the identity matrix. This makes sense, since differentiating and then integrating will not give you the correct constant term.

(c) Explain why there is never a unique solution $p(t)$ to an equation of the form $D(p(t)) = q(t)$. Can you explain using linear algebra concepts?

D cannot be one-to-one (the matrix representing D is 3×4) and so there is never a unique solution to $D(p) = q$. This makes sense, since you can always add a constant term to $p(t)$ and the derivative would be the same.

8. Let $\mathcal{D} = \{d_1, d_2\}$ and $\mathcal{B} = \{b_1, b_2\}$ be bases for vector spaces V, W, respectively. Let $T : V \to W$ be a linear transformation with the property that

$$T(d_1) = 2b_1 - 3b_2 \qquad T(d_2) = -4b_1 + 5b_2$$

Find the matrix for T relative to $\mathcal{D}$ and $\mathcal{B}$.

We compute $T(d_1), T(d_2)$ and write them in the basis $\mathcal{B}$:

$$\begin{pmatrix} 2 & -4 \\ -3 & 5 \end{pmatrix}$$

9. Goodbye leap day...

You will be missed.