# Math 54 Section Worksheet 9

GSI: Jeremy Meza
Office Hours: Monday 3:30-5:30pm, Evans 1047
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## 1 Warm-Up

- Find someone next to you and define what a *basis* of a subspace is. How does a basis allow you to write a vector in a vector space as a column vector in some $\mathbb{R}^n$?
- One of you define what the *kernel* of a linear transformation is. How does the kernel relate to being one-to-one or onto?
- One of you define what the *image* of a linear transformation is. How does the image relate to being one-to-one or onto?

## 2 Problems

For the following problems, we let $\mathbb{P}_n = \{\text{polynomials with real coefficients of degree} \leq n\}$.

1. Let $V = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$.
   (a) Convince yourself that $V$ is a vector space.
   (b) Let $W = \{f \in V \mid f(1) = 0\}$. Show that $W$ is a subspace.
2. Let $p_1(t) = 1 - 3t^2, p_2(t) = 2 + t, p_3(t) = t^3, p_4(t) = 1 + t + t^3$ be polynomials in $\mathbb{P}_3$. Determine whether $p_1, p_2, p_3, p_4$ are linearly independent.
3. Let $V = \{T : \mathbb{R}^2 \to \mathbb{R}^2 \mid T \text{ is linear}\}$ be the set of all linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.
   (a) Is $V$ a vector space? What is addition and scalar multiplication, and do they satisfy the appropriate axioms?
   (b) What do you think is a basis for $V$?
4. Define a linear transformation $T : \mathbb{P}_1 \to \mathbb{R}^2$ by $T(p(t)) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}$. Find polynomials that span the kernel of $T$ and describe the image of $T$. Can you describe the kernel and image of $T$ if we changed the domain to $\mathbb{P}_2$?
5. Let $T : \mathbb{P}_2 \to \mathbb{P}_4$ be the transformation that maps a polynomial $p(t)$ to the polynomial $p(t) + t^2 p(t)$.
   (a) Find the image of $p(t) = 2 - t + t^2$.
   (b) Show that $T$ is a linear transformation.
   (c) Find the matrix for $T$ relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.

6. Let $V = \text{Span}\{\cos t, \sin t\}$.

   (a) Convince yourself that $V$ is a vector space. What is a basis for $V$?

   (b) Define the linear transformation $D : V \to V$ by $D(f) = \frac{\partial f}{\partial t}$. In other words, $D(f)$ is the derivative of $f$. Can you write down a matrix for $D$ by representing functions as column vectors?

7. Let $I : \mathbb{P}_2 \to \mathbb{P}_3$ be the linear transformation defined by

$$I(p(t)) = \int_0^t p(x)\, dx$$

   (a) Find the matrix of $I$ with respect to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$ for $\mathbb{P}_2$ and $\mathbb{P}_3$, respectively.

   (b) Let $D : \mathbb{P}_3 \to \mathbb{P}_2$ be the *derivative* function, defined similarly as in the previous problem. Are $D$ and $I$ inverses? (try computing $DI$ and $ID$).

   (c) Explain why there is never a unique solution $p(t)$ to an equation of the form $D(p(t)) = q(t)$. Can you explain using linear algebra concepts?

8. Let $\mathcal{D} = \{d_1, d_2\}$ and $\mathcal{B} = \{b_1, b_2\}$ be bases for vector spaces $V, W$, respectively. Let $T : V \to W$ be a linear transformation with the property that

$$T(d_1) = 2b_1 - 3b_2 \qquad T(d_2) = -4b_1 + 5b_2$$

   Find the matrix for $T$ relative to $\mathcal{D}$ and $\mathcal{B}$.

9. Goodbye leap day...