Math 54 Section Worksheet 5

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Office Hours: Monday 3:30-5:30pm, Evans 1047
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1 Warm-up

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Choose one of the following:
We say that T is **one-to-one** iff

(a) Every x in $\mathbb{R}^n$ has a unique y in $\mathbb{R}^m$ with $T(x) = y$.

(b) Every x in $\mathbb{R}^n$ has at most one y in $\mathbb{R}^m$ with $T(x) = y$.

(c) For every $y \in \mathbb{R}^m$, there exists an x in $\mathbb{R}^n$ such that $T(x) = y$.

(d) Every y in $\mathbb{R}^m$ has at most one x in $\mathbb{R}^n$ with $T(x) = y$.

We say that T is **onto** iff

(a) Every x in $\mathbb{R}^n$ has at most one y in $\mathbb{R}^m$ with $T(x) = y$.

(b) Every x in $\mathbb{R}^n$ has at least one y in $\mathbb{R}^m$ with $T(x) = y$.

(c) For every y in $\mathbb{R}^m$, there exists an x in $\mathbb{R}^n$ such that $T(x) = y$.

(d) Every y in $\mathbb{R}^m$ has at most one x in $\mathbb{R}^n$ with $T(x) = y$.

2 Problems

1. Suppose $AB = \begin{pmatrix} 5 & 4 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}$. Find A.

2. All matrices below are $n \times n$. Mark each implication as True or False.

   (a) If the equation $Ax = 0$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.

   (b) If the columns of A span $\mathbb{R}^n$, then the columns are linearly independent.

   (c) If A is an $n \times n$ matrix, then the equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$.

   (d) If the equation $Ax = 0$ has a nontrivial solution, then A has fewer than n pivot positions.

   (e) In order for a matrix B to be the inverse of A, both equations $AB = I$ and $BA = I$ must be true.

   (f) If A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB.

(g) If A is an invertible $n \times n$ matrix, then the equation $Ax = b$ is consistent for each b in $\mathbb{R}^n$.

3. Can a square matrix with two identical columns be invertible? Why or why not?

3 Possibly Harder Problems

4. Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $Ax = 0$ has only the trivial solution. Explain why A cannot have more columns than rows.

5. Suppose $AD = I_m$ (the $m \times m$ identity matrix). Show that for any b in $\mathbb{R}^m$, the equation $Ax = b$ has a solution. Explain why A cannot have more rows than columns.

6. Let C be a 2×2 matrix. Prove that there exists 2×2 matrices A, B such that $C = AB - BA$ if and only if $c_{11} + c_{22} = 0$.

7. If A is a 2×1 matrix and B is a 1×2 matrix, prove that $C = AB$ is not invertible.

8. Suppose A is an $n \times n$ matrix. Explain why the columns of A^2 span $\mathbb{R}^n$ whenever the columns of A are linearly independent.

9. (a) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Show that $A^3 = 0$. Compute $(I - A)(I + A + A^2)$.
   What can you conclude about $(I - A)^{-1}$?

   (b) Suppose $A^n = 0$ for some $n > 1$. Find an inverse for $I - A$.