Math 54 Section Worksheet 4

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1 Together

Recall that a *linear transformation* $T : \mathbb{R}^q \to \mathbb{R}^p$ is a function that takes in vectors in $\mathbb{R}^q$ and outputs a unique vector in $\mathbb{R}^p$, such that

$$T(u+v) = T(u) + T(v), \qquad T(cu) = cT(u) \qquad \text{for all } u, v \in \mathbb{R}^q, c \in \mathbb{R}$$

We noted before that every matrix A gives a linear transformation. More specifically, suppose A is a $p \times q$ matrix. Then, we get the linear transformation $T : \mathbb{R}^q \to \mathbb{R}^p$ such that $T(v) = Av$.

Example 1.1. Describe in words the transformation that is given by the matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Example 1.2. Let's consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by flipping around the y axis.

In fact, it holds that every linear transformation is given by some matrix transformation. That is, if $T : \mathbb{R}^q \to R^p$ is a linear transformation, then we will construct a matrix A such that $T(x) = Ax$ for all $x \in R^q$.

2 Problems

1. Find the matrix associated to the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by flipping around the yz-plane and then projecting to the xy-plane.
2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that reflects each vector $\mathbf{x} = (x_1, x_2, x_3)$ through the plane $x_3 = 0$ onto $T(\mathbf{x}) = (x_1, x_2, -x_3)$. Show that T is a linear transformation.

3. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $y_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, and $y_2 = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps e_1 to y_1 and maps e_2 to y_2. Find the images of $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

4. Mark each statement True or False.

   (a) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.

   (b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors about the origin through an angle φ, then T is a linear transformation.

   (c) When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.

   (d) A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector x in $\mathbb{R}^n$ maps onto some vector in $\mathbb{R}^m$.

   (e) If A is a 3×2 matrix, then the transformation $x \mapsto Ax$ cannot be one-to-one.

   (f) Not every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.

   (g) The columns of the standard matrix for a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the images of the columns of the $n \times n$ identity matrix.

   (h) The standard matrix of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects points through the horizontal axis, the vertical axis, or the origin has the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, where a and d are ± 1.

   (i) A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.

   (j) If A is a 3×2 matrix, then the transformation $x \mapsto Ax$ cannot map $\mathbb{R}^2$ onto $\mathbb{R}^3$.

5. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one. Describe the possible reduced row echelon forms of its corresponding matrix.

6. Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is onto. Describe the possible reduced row echelon forms of its corresponding matrix.