Math 54 Section Worksheet 3

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Office Hours: Monday 3:30-5:30pm, Evans 1047
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1 Warm-Up

1. Describe the solutions of the following system in parametric vector form. Give a geometric description of the solution set.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}$$

2. Which of the following are correct?
   (a) I can multiply an $m \times n$ matrix by an $m \times 1$ matrix and I get a $1 \times n$ matrix.
   (b) I can multiply an $m \times n$ matrix by an $n \times 1$ matrix and I get an $m \times 1$ matrix.
   (c) I can multiply an $m \times n$ matrix by an $m \times 1$ matrix and I get an $n \times 1$ matrix.
   (d) I can multiply an $m \times n$ matrix by an $1 \times n$ matrix and I get a $1 \times m$ matrix.

2 Problems

1. Let $v_1 = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ 10 \\ 6 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ -9 \\ h \end{pmatrix}$. For what values of h is v_3 in $\text{Span}\{v_1, v_2\}$, and for what values of h is $\{v_1, v_2, v_3\}$ linearly dependent?

2. Mark each statement True or False.
   (a) The columns of a matrix A are linearly independent if the equation $Ax = 0$ has the trivial solution.
   (b) If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.
   (c) The columns of any 4×5 matrix are linearly dependent.
   (d) If x and y are linearly independent, and if $\{x, y, z\}$ is linearly dependent, then z is in $\text{Span}\{x, y\}$.
   (e) Two vectors are linearly dependent if and only if they lie on a line through the origin.

(f) If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.

(g) If x and y are linearly independent, and if z is in $\text{Span}\{x, y\}$, then $\{x, y, z\}$ is linearly dependent.

(h) If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more vectors than there are entries in each vector.

3. For the given cases below, (i) does the equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution and (ii) does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for every possible $\mathbf{b}$?

   (a) A is a 3×3 matrix with three pivot positions.

   (b) A is a 3×3 matrix with two pivot positions.

   (c) A is a 3×2 matrix with two pivot positions.

   (d) A is a 2×4 matrix with two pivot positions.

4. For the statements below, mark either True or False. If True, give a justification. If False, provide a counterexample.

   (a) If $v_1, \dots, v_4$ are in $\mathbb{R}^4$ and $v_3 = 0$, then $\{v_1, \dots, v_4\}$ is linearly dependent.

   (b) If v_1 and v_2 are in $\mathbb{R}^4$ and v_2 is not a scalar multiple of v_1, then $\{v_1, v_2\}$ is linearly independent.

   (c) If $v_1, \dots, v_4$ are linearly independent vectors in $\mathbb{R}^4$, then $\{v_1, v_2, v_3\}$ is also linearly independent.

3 Possibly Harder Problems

5. Suppose A is a 3×3 matrix and b is a vector in $\mathbb{R}^3$ with the property that $Ax = b$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^3$.

6. Suppose $A\mathbf{x} = \mathbf{b}$ has a solution. Prove that the solution is unique if and only if the matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. (recall that the "trivial solution" is the solution $\mathbf{x} = 0$.)

7. Construct a 2×2 matrix A such that the solution set of the equation $A\mathbf{x} = \mathbf{0}$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b}$ in $\mathbb{R}^2$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ is not a line in $\mathbb{R}^2$ parallel to the solution set of $A\mathbf{x} = \mathbf{0}$.