Math 54 Section Worksheet 2

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Office Hours: Monday 3:30-5:30pm, Evans 1047
Friday, January 31, 2020

1 Warm-Up

- Turn to someone next to you and define to them what a *linear combination* is. What type of object is it? How do you correctly use the term in a sentence?
- Now have the other person define to you what *span* is. What type of object is it? How do you correctly use the term in a sentence?
- Multiply the following:

$$\begin{pmatrix} 1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 10 & 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 0 \\ -5 \end{pmatrix}$$

2 Together

Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$, and $b = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$. Let's determine if $\mathbf{b}$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Now, let's solve the system:

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 4 \\ 5x_2 + 4x_3 &= 3 \\ 2x_1 + x_2 + 1x_3 &= 7 \end{aligned}$$

Notice any similarities?

3 Problems

1. Mark each statement True or False.

   (a) The weights $c_1, \ldots, c_p$ in a linear combination $c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p}$ cannot all be zero.

   (b) When $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ contains the line through $\mathbf{u}$ and the origin.

   (c) Asking whether the linear system corresponding to an augmented matrix $\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b}\end{bmatrix}$ has a solution amounts to asking whether $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

   (d) If $m \neq n$, the columns of an $m \times n$ matrix A can span $\mathbb{R}^n$.

   (e) If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$, then the equation $Ax = b$ has a solution for all $b \in \mathbb{R}^m$.

   (f) If the equation $Ax = b$ has a solution for all $b \in \mathbb{R}^m$, then the columns of an $m \times n$ matrix span $\mathbb{R}^m$.

2. Let A be a 3×2 matrix. Explain why the equation $Ax = b$ cannot be consistent for all b in $\mathbb{R}^3$. Generalize your argument to the case of an arbitrary A with more rows than columns.

3. Describe the solutions of the following system in parametric vector form. Give a geometric description of the solution set.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}$$

4. Let $v_1 = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ 10 \\ 6 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ -9 \\ h \end{pmatrix}$. For what values of h is v_3 in $\text{Span}\{v_1, v_2\}$?

4 Possibly Harder Problems

5. Suppose A is a 3×3 matrix and b is a vector in $\mathbb{R}^3$ with the property that $Ax = b$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^3$.

6. Suppose $A\mathbf{x} = \mathbf{b}$ has a solution. Prove that the solution is unique if and only if the matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. (recall that the "trivial solution" is the solution $\mathbf{x} = 0$.)

7. Construct a 2×2 matrix A such that the solution set of the equation $A\mathbf{x} = \mathbf{0}$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b}$ in $\mathbb{R}^2$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ is not a line in $\mathbb{R}^2$ parallel to the solution set of $A\mathbf{x} = \mathbf{0}$.