Math 54 Section Worksheet 17

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Friday April 17, 2020

1 Warm-Up

1. Write down an example of a *second-order*, *homogenous*, *linear*, *constant coefficient* differential equation. Private chat with someone and explain to them what all the terms above mean.

2. Solve the differential equation

$$y' + ky = 0$$

with initial value $y(0) = 5$.

2 Virtually Together

We want to solve the differential equation

$$ay'' + by' + cy = 0 \tag{1}$$

Before we actually solve this, let's abstract the problem to see if we can connect it to linear algebra.

Let's consider the vector space $V = \{\text{smooth functions } f : \mathbb{R} \to \mathbb{R}\}$. Define a transformation $D : V \to V$ by $D(f) = af'' + bf' + cf$.

Problem: Show that D is a linear transformation.

Finding solutions $y(t)$ to (1) is equivalent to finding elements of...

(Circle one) (a) $\ker(D)$ (b) $\mathrm{Im}(D)$ (c) V (d) $\mathrm{Nul}(D)^{\perp}$.

This means that if we want to describe all the solutions to (1), we want to find a basis for a null space!

Fact: Because (1) is 2nd-order, the space of solutions will be a 2-dimensional subspace inside V.

Thus, our null space is 2-dimensional, and so we just need to find 2 linearly independent solutions and any other solution will simply be some linear combination of them. So, how do we find just 2 linearly independent solutions to a differential equation? We guess, and then we check.

Example 2.1. Let's solve the equation $y'' - 5y' + 6y = 0$.
Guess: $y(t) = e^{rt}$.
Check: Plugging in, we get $r^2e^{rt} - 5re^{rt} + 6e^{rt} = 0$ and hence

$$r^2 - 5r + 6 = 0 \implies (r-3)(r-2) = 0 \implies r = 3, 2$$

We claim that 2 linearly independent solutions are $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$.

If the *characteristic equation* in the above example turned out to have a double root or complex roots, then we amend our 2 linearly independent solutions:

	Two linearly independent solutions
2 distinct real roots r_1, r_2	$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}$
1 double root $r_1 = r_2 = r$	$y_1(t) = e^{rt}, \quad y_2(t) = te^{rt}$
2 complex roots $r_1 = \alpha + \beta i$ $r_2 = \alpha - \beta i$	$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}$ OR $y_1(t) = e^{\alpha t}\cos(\beta t), \quad y_2(t) = e^{\alpha t}\sin(\beta t)$

Now, what exactly does it mean for solutions to be linearly independent? We will talk about this more another time, but for now...

3 Problems

1. Find a general solution to the given differential equation.
   (a) $y'' - y' - 11y = 0$
   (b) $4y'' - 4y' + y = 0$
   (c) $z'' - 6z' + 10z = 0$
2. Find a particular solution to the given initial value problem.
   (a) $y'' - 4y' - 5y = 0; y(-1) = 3, y'(-1) = 9$
   (b) $y'' + 2y' + y = 0; y(0) = 1, y'(0) = -3$
   (c) $w'' - 2w' + 4w = 0; w(0) = 0, w'(0) = 1$

4 Challenge

3. Let y_1, y_2 be two solutions to $y'' + py' + qy = 0$. Define the *Wronskian* $W[y_1, y_2]$ to be the determinant of the matrix $\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$.
   (a) Show that $W[y_1, y_2]$ satisfies the differential equation $W' + pW = 0$.
   (b) Solve the separable equation in part (a).
   (c) Conclude that the Wronskian is either identically 0 or never 0.