Math 54 Section Worksheet 16

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1 Virtually Together

Let's first recall the definition of an orthogonal matrix:

Definition 1.1. Let U be an $n \times n$ matrix. Then, U is *orthogonal* iff $U^T U = I$.

Note that since U is square, we also have that $UU^T = I$. Unwrapping this definition, we can see that the columns of U are ____________________________. Now, what's so important about these matrices? A good start is listing some properties they satisfy:

Proposition 1.1. *Let U be an $n \times n$ orthogonal matrix and $x, y \in \mathbb{R}^n$. Then,*

1. $\|Ux\| =$
2. $Ux \cdot Uy =$

What this says is that, viewed as a linear transformation, U doesn't change the lengths of vectors or angles between any 2 vectors. Do you have a picture in your mind? It might look something like the following:

In fact, geometrically, *every* orthogonal matrix is just some rotation or reflection! (See the challenge problems). In general, suppose U is an $m \times n$ matrix with orthonormal columns. Note that if $m \neq n$, then U is NOT an orthogonal matrix, because by definition an orthogonal matrix must be square. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ denote the columns of U and let $\mathbf{x} \in \mathbb{R}^n$, then

$$U^T U \mathbf{x} = \rule{8cm}{0.4pt}$$

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For this reason, we call UU^T a *projection matrix*, since it maps vectors $\mathbf{x}$ to $\text{proj}_{\text{Col}\, U}\, \mathbf{x}$.[1]

Switching topics, let's explain the following equalities:

$\dim \text{Col}\, A + \dim \text{Nul}\, A = n$ because ____

$\dim \text{Row}\, A + \dim \text{Nul}\, A = n$ because ____

$\dim \text{Row}\, A + \dim \text{Nul}\, A^T = m$ because ____

$\dim \text{Col}\, A + \dim \text{Nul}\, A^T = m$ because ____

Thus,

$$\dim \text{Col}\, A = \dim \text{Row}\, A$$

[1] In general, a projection matrix P need not be an orthogonal projection. Instead, it only needs to satisfy $P^2 = P$. If P is also an orthogonal projection, then the further condition $P^2 = P = P^T$ holds.

2 Problems

1. Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$. What spaces do $\operatorname{Col} A, \operatorname{Row} A, \operatorname{Nul} A, \operatorname{Nul} A^T$ lie in? Find bases for each one, and then try to draw a picture of $\mathbb{R}^2$ and $\mathbb{R}^3$ with each drawn in.

2. Let $P = UU^T$, where U has orthonormal columns.

   (a) Show that $P^2 = P$.

   (b) Show that P only has eigenvalues 0, 1.

   (c) What linear transformation does P correspond to geometrically? Can you think of eigenvectors of P with eigenvalues 0, 1?

3. True or False?

   (a) A matrix with orthonormal columns is an orthogonal matrix.

   (b) If the columns of an $m \times n$ matrix A are orthonormal, then the linear mapping $\mathbf{x} \mapsto A\mathbf{x}$ preserves lengths.

   (c) An orthogonal matrix is invertible.

   (d) If x, y are orthogonal vectors in $\mathbb{R}^n$ and U is an orthogonal matrix, then $(Ux) \cdot y = 0$.

   (e) If the columns of an $n \times p$ matrix U are orthonormal, then $UU^T\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of U.

   (f) If an $n \times p$ matrix U has orthonormal columns, then $UU^Tx = x$ for all x in $\mathbb{R}^n$.

   (g) If $P = UU^T$ where U has orthonormal columns, then P is invertible.

3 Challenge

4. Let A be a real orthogonal $n \times n$ matrix. Prove that every eigenvalue of A is either 1 or -1, and also that $\det A = \pm 1$. (Hint: for the former, look at $Av \cdot v$ for a chosen v; for the latter, look at the characteristic polynomial).

5. Let A be a 2×2 orthogonal matrix. Prove that the linear transformation $x \mapsto Ax$ is either a rotation about the origin or a reflection about a line through the origin. (Hints: Write out an arbitrary 2×2 matrix and find some equations the entries must satisfy owing to orthogonality and the problem above. Then, see what A does to e_1 and e_2).

6. Let A be a real orthogonal $n \times n$ matrix. Prove that if n is odd, then A has at least one real eigenvalue. (Hint: look at the characteristic polynomial and use some property of continuous functions).

7. Let A be a 3×3 orthogonal matrix.

   (a) Use some problems above to deduce that A has a real eigenvector with eigenvalue either 1 or -1.

   (b) Suppose there is some vector v such that $Av = v$. Make a guess as to what the linear transformation $x \mapsto Ax$ can be, geometrically, as in the problem above. (What does the vector v represent?)

   (c) Suppose there is a vector v such that $Av = -v$. Make a guess as to what the linear transformation $x \mapsto Ax$ can be, geometrically, as in the problem above.