Math 54 Section Worksheet 14 Solutions

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1 Warm-Up

True or False?

T (a) For a square matrix A, vectors in $\operatorname{Col} A$ are orthogonal to vectors in $\operatorname{Nul} A^T$.

F (b) The orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto a subspace W can sometimes depend on the orthogonal basis for W used to compute $\hat{\mathbf{y}}$.

F (c) A space W has only finitely many orthonormal bases.

2 Virtually Together

Let's recall how to compute an orthogonal projection. Let $\mathbf{v} \in \mathbb{R}^n$ and $W \subseteq \mathbb{R}^n$ be a subspace, say of dimension d. The *orthogonal projection* of $\mathbf{v}$ onto W is given by

$$\operatorname{proj}_W(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 + \cdots + \frac{\mathbf{v}\cdot\mathbf{w}_n}{\mathbf{w}_n\cdot\mathbf{w}_n}\mathbf{w}_n$$

where $\{\mathbf{w}_1, \ldots, \mathbf{w}_d\}$ is an orthogonal basis for W.

What happens when we have a subspace W but are not given an orthogonal basis for W? How do we compute the orthogonal projection then? The answer is... we start with any old basis of W and from it construct a new, orthogonal basis of W using the **Gram-Schmidt algorithm**.

Let's find an orthogonal basis for the column space of the matrix:

$$A = \begin{pmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{pmatrix}$$

3 Problems

1. Let $A = \begin{pmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$.

   (a) Use the Gram-Schmidt process to find an orthogonal basis for Col A.
   Label the columns of A b_1, b_2, b_3, respectively.

   $$w_1 = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \qquad w_2 = b_2 - \frac{b_2 \cdot w_1}{w_1 \cdot w_1} w_1 = \begin{pmatrix} 6 \\ -8 \\ -2 \\ -4 \end{pmatrix} - \frac{-36}{12} \begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

   $$w_3 = b_3 - \frac{b_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{b_3 \cdot w_2}{w_2 \cdot w_2} w_2 = \begin{pmatrix} 6 \\ 3 \\ 6 \\ -3 \end{pmatrix} - \frac{6}{12} \begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix} - \frac{30}{12} \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 3 \\ -1 \end{pmatrix}$$

   (b) Compute the orthogonal projection of $\mathbf{v}$ onto Col A.

   $$\text{proj}_{\text{Col}\, A}\, \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \mathbf{w}_1 + \frac{\mathbf{v} \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} \mathbf{w}_2 + \frac{\mathbf{v} \cdot \mathbf{w}_3}{\mathbf{w}_3 \cdot \mathbf{w}_3} \mathbf{w}_3 = \frac{12}{12} \begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix} + \frac{12}{12} \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix} + 0 = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 0 \end{pmatrix}$$

   (c) Compute $A^T(\mathbf{v} - \text{proj}_{\text{Col}\, A}\, \mathbf{v})$. What should you get? 0.

   (d) Solve the linear system $A\mathbf{x} = \text{proj}_{\text{Col}\, A}\, \mathbf{v}$. $\mathbf{x} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$

   (e) Solve the linear system $A^T A\mathbf{x} = A^T\mathbf{v}$. Compare your answer to the part above. For your convenience, $A^T A = \frac{1}{6} \begin{pmatrix} 2 & -6 & 6 \\ -6 & 20 & 2 \\ 1 & 2 & 15 \end{pmatrix}$. You should get the same answer as in part (d).

2. True or False?

   T (a) The Gram-Schmidt process produces from a linearly independent set $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ an orthogonal set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ with the property that for each k, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span the same subspace as that spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_k$.

   F (b) If W is a subspace with basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is another basis of W that is also orthogonal, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ was gotten from $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ by Gram-Schmidt.

   F (c) If L is a line through 0 and if $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto L, then $\|\hat{\mathbf{y}}\|$ gives the distance from $\mathbf{y}$ to L.

   T (d) For each $\mathbf{y}$ and each subspace W, the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$ is orthogonal to W.

   T (e) If $\mathbf{y}$ is in a subspace W, then the orthogonal projection of $\mathbf{y}$ onto W is $\mathbf{y}$ itself.

   T (f) If W is a subspace of $\mathbb{R}^n$ and if $\mathbf{v}$ is in both W and $W^\perp$, then $\mathbf{v}$ must be the zero vector.

   T (g) If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ is in a subspace of W and $\mathbf{z}_2$ is in $W^\perp$, then $\mathbf{z}_1$ must be the orthogonal projection of $\mathbf{y}$ onto W.

   F (h) The best approximation to $\mathbf{y}$ by elements of a subspace W is given by the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$.

   T (i) If $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent and if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.