Math 54 Section Worksheet 14

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1 Warm-Up

True or False?

1. For a square matrix A, vectors in $\operatorname{Col} A$ are orthogonal to vectors in $\operatorname{Nul} A^T$.
2. The orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto a subspace W can sometimes depend on the orthogonal basis for W used to compute $\hat{\mathbf{y}}$.
3. A space W has only finitely many orthonormal bases.

2 Virtually Together

Let's recall how to compute an orthogonal projection. Let $\mathbf{v} \in \mathbb{R}^n$ and $W \subseteq \mathbb{R}^n$ be a subspace, say of dimension d. The *orthogonal projection* of $\mathbf{v}$ onto W is given by

$$\operatorname{proj}_W(\mathbf{v}) =$$

where $\{\mathbf{w}_1, \ldots, \mathbf{w}_d\}$ is an ____________________________ for W.

What happens when we have a subspace W but are not given an orthogonal basis for W? How do we compute the orthogonal projection then? The answer is... we start with any old basis of W and from it construct a new, orthogonal basis of W using the **Gram-Schmidt algorithm**.

Let's find an orthogonal basis for the column space of the matrix:

$$A = \begin{pmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{pmatrix}$$

3 Problems

1. Let $A = \begin{pmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$.

   (a) Use the Gram-Schmidt process to find an orthogonal basis for $\operatorname{Col} A$.

   (b) Compute the orthogonal projection of $\mathbf{v}$ onto $\operatorname{Col} A$.

   (c) Compute $A^T(\mathbf{v} - \operatorname{proj}_{\operatorname{Col} A} \mathbf{v})$. What should you get?

   (d) Solve the linear system $A\mathbf{x} = \operatorname{proj}_{\operatorname{Col} A} \mathbf{v}$.

   (e) Solve the linear system $A^T A\mathbf{x} = A^T\mathbf{v}$. Compare your answer to the part above. For your convenience, $A^T A = \frac{1}{6}\begin{pmatrix} 2 & -6 & 6 \\ -6 & 20 & 2 \\ 1 & 2 & 15 \end{pmatrix}$.

2. True or False?

   (a) The Gram-Schmidt process produces from a linearly independent set $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ an orthogonal set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ with the property that for each k, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span the same subspace as that spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_k$.

   (b) If W is a subspace with basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is another basis of W that is also orthogonal, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ was gotten from $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ by Gram-Schmidt.

   (c) If L is a line through 0 and if $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto L, then $\|\hat{\mathbf{y}}\|$ gives the distance from $\mathbf{y}$ to L.

   (d) For each $\mathbf{y}$ and each subspace W, the vector $\mathbf{y} - \operatorname{proj}_W \mathbf{y}$ is orthogonal to W.

   (e) If $\mathbf{y}$ is in a subspace W, then the orthogonal projection of $\mathbf{y}$ onto W is $\mathbf{y}$ itself.

   (f) If W is a subspace of $\mathbb{R}^n$ and if $\mathbf{v}$ is in both W and $W^\perp$, then $\mathbf{v}$ must be the zero vector.

   (g) If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ is in a subspace of W and $\mathbf{z}_2$ is in $W^\perp$, then $\mathbf{z}_1$ must be the orthogonal projection of $\mathbf{y}$ onto W.

   (h) The best approximation to $\mathbf{y}$ by elements of a subspace W is given by the vector $\mathbf{y} - \operatorname{proj}_W \mathbf{y}$.

   (i) If $W = \operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent and if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.