Math 54 Section Worksheet 13 Solutions

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1 Warm-Up

1. Let $\mathbf{a} = (1, -2, 3)^T$ and $\mathbf{b} = (4, 1, -1)^T$. Calculate $\mathbf{a} \cdot \mathbf{b}$. Calculate the *norm* $\|\mathbf{a}\|$.

2. Try to recall the following concepts:

orthogonal	orthonormal	orthogonal projection	orthogonal complement

2 Virtually Together

Let A be an $n \times n$ matrix. Recall that if A is *diagonalizable*, then we can find a basis for $\mathbb{R}^n$ consisting solely of eigenvectors of A. By doing this, we can take the linear transformation $x \mapsto Ax$ and, when we write the matrix of this transformation in this basis of eigenvectors, we get a *diagonal* matrix.

The upshot of this is that if we want to compute Ax, we can write x in the basis of eigenvectors, and then just multiply that column vector by a diagonal matrix. Diagonal matrices are easy to multiply!

Unfortunately, sometimes it's not so easy to write a vector in a certain basis. For example, if $\mathbf{x}$ is a vector that we are trying to write in a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then to find $[\mathbf{x}]_{\mathcal{B}}$, we need to solve for the coefficients $c_1, \ldots, c_n$ in the linear combination

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$$

which means we need to solve a linear system!

This takes some time. Or does it? Sometimes, when the basis has a very special quality that the standard basis has, but that not every basis has, solving for these coefficients is quite easy. Can you guess what it is?

That's right, you guessed it! It's orthogonality.

Let's say $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is an orthonormal set of vectors in $\mathbb{R}^n$.

(a) Is this set of vectors linearly independent? Yes.

(b) Does this set form a basis for $\mathbb{R}^n$? Yes; they are n linearly independent vectors.

(c) Show that for any vector $\mathbf{v}$ in $\mathbb{R}^n$, we can expand

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{b}_1)\mathbf{b}_1 + \cdots + (\mathbf{v} \cdot \mathbf{b}_n)\mathbf{b}_n$$

(d) What happens when $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is not orthonormal, but just orthogonal? Does this make much of a difference? Not much of a difference, we just divide each $(\mathbf{v} \cdot \mathbf{b}_i)$ by $(\mathbf{b}_i \cdot \mathbf{b}_i)$.

Our conclusion is that it's very easy to expand a vector as a linear combination of basis vectors *if* the basis vectors are orthogonal.

3 Problems

1. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Find the eigenvalues and eigenspaces of A. Take 2 eigenvectors, one from each eigenspace, and take their dot product. What do you get? Stay tuned in 2 weeks when we talk about *symmetric* matrices.

   The eigenvalues are $\lambda = 3, -1$, with eigenspaces $\text{Span}\{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\}, \text{Span}\{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)\}$, respectively. The dot product between the eigenvectors is 0. They are orthogonal!

2. Let $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(a) Find the eigenvalues and eigenspaces of A. There is only the eigenvalue 0, with eigenspace $\text{Span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$.

(b) Find the eigenvalues and eigenspaces of B. There is only the eigenvalue 0, with eigenspace $\text{Span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}$.

(c) (Tricky) Prove that A and B are not similar. (Hint: try to find P).

3. True or False?

T (a) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.

T (b) For any scalar c, $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$.

T (c) $\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0$.

F (d) For any scalar c, $\|c\mathbf{v}\| = c\|\mathbf{v}\|$.

T (e) If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance from $\mathbf{u}$ to $-\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

T (f) If vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span a subspace W and if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \ldots, p$, then $\mathbf{x}$ is in $W^\perp$.

T (g) If $\mathbf{x}$ is orthogonal to every vector in a subspace W, then $\mathbf{x}$ is in $W^\perp$.

F (h) If $\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

F (i) For a square matrix A, vectors in $\operatorname{Col} A$ are orthogonal to vectors in $\operatorname{Nul} A$.

T (j) Not every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.

F (k) Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.

T (l) If $\mathbf{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix

F (m) If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.

F (n) If L is a line through 0 and if $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto L, then $\|\hat{\mathbf{y}}\|$ gives the distance from $\mathbf{y}$ to L.

F (o) If a set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ has the property that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$, then S is an orthonormal set.

T (p) For each $\mathbf{y}$ and each subspace W, the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$ is orthogonal to W.

F (q) The orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto a subspace W can sometimes depend on the orthogonal basis for W used to compute $\hat{\mathbf{y}}$.

T (r) If $\mathbf{y}$ is in a subspace W, then the orthogonal projection of $\mathbf{y}$ onto W is $\mathbf{y}$ itself.

T (s) If W is a subspace of $\mathbb{R}^n$ and if $\mathbf{v}$ is in both W and $W^\perp$, then $\mathbf{v}$ must be the zero vector.

T (t) In the Orthogonal Decomposition Theorem, each term $\frac{\mathbf{y} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i}\mathbf{u}_i$ in the formula for $\hat{\mathbf{y}}$ is itself an orthogonal projection of $\mathbf{y}$ onto a subspace of W.

T (u) If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ is in a subspace of W and $\mathbf{z}_2$ is in $W^\perp$, then $\mathbf{z}_1$ must be the orthogonal projection of $\mathbf{y}$ onto W.

F (v) The best approximation to $\mathbf{y}$ by elements of a subspace W is given by the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$.

T (w) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for W, then multiplying $\mathbf{v}_3$ by a scalar c gives a new orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, c\mathbf{v}_3\}$.

T (x) If $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent and if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.

T (y) If $\mathbf{x}$ is not in a subspace W, then $\mathbf{x} - \text{proj}_W \mathbf{x}$ is not zero.

snaps emoji (z) 2020 can chill tf out...