Math 54 Section Worksheet 12 Solutions

GSI: Jeremy Meza
Office Hours: Mon 3:30-5:30pm, Zoom ID: 7621822286
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1 Virtually Together

Let's first start with abstract vector spaces... let's find the eigenvalues and corresponding eigenspaces of the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ given by $T(p(t)) = t\frac{\partial}{\partial t}p(t)$.

By definition, an eigenvalue λ and an eigenvector $p(t)$ for T satisfy $T(p(t)) = \lambda p(t)$. Unwrapping this, we see that $t\frac{\partial}{\partial t}p(t) = \lambda p(t)$. This is a differential equation, one in which we could solve (since it is *separable*, a term we will learn about later). However, we will find the eigenvalues and eigenvectors using a little more linear algebra...

Let's first write a matrix associated to T. In the basis $\{1, t, t^2\}$, T is represented as $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Right away, we see an eigenbasis for T, since this matrix is already diagonal! The eigenvalues are $0, 1, 2$ with eigenspaces $\text{Span}(e_1), \text{Span}(e_2), \text{Span}(e_3)$, respectively, in the basis $\{1, t, t^2\}$. Converting these back to polynomials, we have $\text{Span}(1), \text{Span}(t), \text{Span}(t^2)$ as the eigenspaces.

Now let's get to the complex stuff... let's find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}$.

The eigenvalues are the roots of

$$(5 - \lambda)(3 - \lambda) + 2 = \lambda^2 - 8\lambda + 17$$

and so $\lambda = \frac{8 \pm \sqrt{64 - 4 \cdot 17}}{2} = 4 \pm i$. The eigenspace for $\lambda = 4 + i$ is found by finding the nullspace of $\begin{pmatrix} 1 - i & -2 \\ 1 & -1 - i \end{pmatrix}$. We could row-reduce, but actually these rows are just (complex) multiples of each other. Indeed, if I multiply the second row by $1 - i$, I get the first row (since $(1 - i)(-1 - i) = -2$). Thus, looking at the second row, I see that the eigenspace is $\text{Span}\left\{\begin{pmatrix} 1 + i \\ 1 \end{pmatrix}\right\}$. When you row-reduce, you might find that the eigenspace is something like $\text{Span}\left\{\begin{pmatrix} 2 \\ 1 - i \end{pmatrix}\right\}$, for example. But fear not, these are the same space! Finally, the eigenspace for $\lambda = 4 - i$ is found by taking the span of the complex conjugate of an eigenvector for $\lambda = 4 + i$.

2 Problems

1. Let $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

   (a) Find the eigenvalues and eigenvectors of A. Is A diagonalizable?

   (b) Find the eigenvalues and eigenvectors of B. Is B diagonalizable?

   (c) (Tricky) Find a matrix P such that $A = PBP^{-1}$.

   The eigenvalues of both A and B are $\lambda = 2$. The eigenspace for A is $\text{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ and the eigenspace for B is $\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

   You can find the matrix P by looking at the equation $AP = PB$, putting some indeterminants in for the entries of P, and solving. You will find $P = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$.

2. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Find the eigenvalues of A and for each eigenvalue, find a basis for the corresponding eigenspace. Is A diagonalizable?

   The eigenvalues are $\lambda = \pm i$. The eigenspace for $\lambda = i$ is found by solving for x in $\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} x = 0$, from which we see that the eigenspace is $\text{Span}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix}\right\}$. The eigenspace for $\lambda = -i$ is the complex conjugate of the eigenspace above, $\text{Span}\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}\right\}$.

3. Let $A = \begin{pmatrix} 1 & 1 & -5 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}$. Find the eigenvalues of A. For each eigenvalue, find a basis for the corresponding eigenspace. Is A diagonalizable?

   The eigenvalues are $\lambda = 2, 2 \pm 2i$. There are 3 distinct eigenvalues, so A is diagonalizable. The eigenspaces are $\text{Span}\left\{\begin{pmatrix} -11 \\ 4 \\ 3 \end{pmatrix}\right\}$, $\text{Span}\left\{\begin{pmatrix} -1+2i \\ 0 \\ 1 \end{pmatrix}\right\}$, and $\text{Span}\left\{\begin{pmatrix} -1-2i \\ 0 \\ 1 \end{pmatrix}\right\}$, respectively.

4. All matrices below are $n \times n$. True or False?

   T (a) If A is similar to B, and both are invertible, then A^{-1} is similar to B^{-1}.

   F (b) If A and B have the same eigenvalues, then A is similar to B.

   F (c) If A and B have the same characteristic polynomial, then A is similar to B.

F (d) If A and B are both diagonalizable, then A is similar to B.

F (e) If A and B are both diagonalizable and have the same eigenvalues, then A is similar to B.

T (f) If A and B are both diagonalizable and have the same characteristic polynomial, then A is similar to B.

F (g) If all the eigenvalues of A are 0, then A is the 0 matrix.

T (h) If A is a real matrix with eigenvalue $a + bi$, then $a - bi$ is also an eigenvalue of A.

T (i) If A is a real matrix with eigenvector $\mathbf{u} + i\mathbf{v}$, then $\mathbf{u} - i\mathbf{v}$ is also an eigenvector of A.

T (j) The matrix A always has n complex eigenvalues, counting multiplicity.

T (k) If A is diagonalizable, then A^2 is diagonalizable.

F (l) If A^2 is diagonalizable, then A is diagonalizable.

3 Extra (possibly difficult) Problems

5. Prove that any matrix of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ where $a, b \in \mathbb{R}$ with $b \neq 0$, is not diagonalizable.

6. Let V be the vector space $\text{Span}\{\cos t, \sin t\}$. Let $D : V \to V$ be the function $D(f) = \frac{\partial}{\partial t} f$.

   (a) Write down a matrix that represents D.

   (b) Can you find an alternate basis $\mathcal{E}$ so that when written in this basis, D is a diagonal matrix?

   Hint: You might find the following identities useful:

   $$e^{it} = \cos t + i \sin t \qquad e^{-it} = \cos t - i \sin t$$

7. Diagonalize $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.